On Virgil: My Opening Lecture to Mathematics 120

PETER TAYLOR

My name is Peter Taylor and this is Mathematics 120 — a course in differential and integral calculus designed for students in any program who wish to study some mathematics. The main question for you to answer right now is whether you ought to take this course, so I propose to spend the next hour helping you make that decision.

If you take this course you will have to work very hard That's all right; none of us objects to hard work But in your case this work will be extremely frustrating, certainly at the beginning Again and again you will come to me in exasperation and cry out *Why can't I do this?* And I will answer, *I don't know*, and then add hopefully, *It's just mathematics. How long have you been studying mathematics?* You will pause briefly and answer *Thirteen years* *Thirteen years?* I will echo in astonishment, *That is a long time Then why can't I do your problems?*

The sad truth is that so much of that thirteen years was spent so badly, that your experience counts for almost nothing. So much of what you did was so tedious that your perceptions are now dulled, your mind has lost its keen edge and your mathematical fireplace is cold and damp. That is why you can't do my problems

The trouble is that you have spend almost all your time copying the works of others, like the school boy who is detained to copy 200 lines of Latin poetry. A sad and mindless task that, especially if he knows almost no Latin Guaranteed to turn him bitterly against the language.

Of course it might have been Virgil. I can imagine copying Virgil under some circumstances. I can certainly imagine reading him, preferably aloud The Aeneid is a masterpiece.

ARMA virumque cano, Troiae qui primus ab oris
Italian fato profugus Laviniaque venit
litora, multum ille et terris iactatus et alto
vi superum, saevae memorem Iunonis ob iram,
multa quoque et bello passus, dum conderet urbem
inferretque deos Latio; genus unde Latinum
Albanique patres atque altae moenia Romae

Good stuff that, though it takes a bit of practice.

But the texts *you* have copied so meticulously year after year are not masterpieces For the most part they are tedious tales by minor Latin poets. The meter is poor and the stories are contrived. They describe obscure battles or ill-conceived skirmishes, where victory comes not by any stroke of genius, but from some unseen technical advantage enjoyed by the other side.

Your teachers chose this carefully-edited pulp over Virgil because they thought it would be easier to digest That may be so, but it has ruined your appetite

You would have been better to write your own poetry The trouble is, you know hardly any Latin You can't read it very well, and you can't speak it at all You never acquired any burning desire to master it because you weren't exactly turned on by the pathetic examples that were put before you.

My first task then is to restore your appetite for Latin poetry I shall do this by reading the Aeneid Not for you to copy however; not even Virgil should in fact be copied. I shall read aloud in as rich and melodic a voice as I can master, and you must sit back and let the stately hexameter capture your soul, and the heroic tale set it on fire.

And then I shall read you a few modest poems of my own that have been inspired by my encounters with Virgil. They are not great like the Aeneid, but I have worked on them very hard and they are the best that I can do In spite of their modesty they have an important place in this classroom because I am your teacher

Then finally you must find the muse yourselves. At that time you will be glad that I shared my poems with you. For Virgil is a hard act to follow

What do you think? Are you cowed or are you outraged? Have I been unfair? Have I even been serious? Perhaps I have been jesting. There, a few weak smiles are appearing at the front Others are still sitting rigid with tight faces, a hint of panic in the corners of their eyes. *My God, mother was right, I should have taken Sociology after all.* That's better, smiles all round now A general easing of tension. A respite well earned. But only a respite The hour is not yet over, and I have more to say Indeed I have barely begun

I would talk more about Latin poetry, but I fear some of you may find the metaphor difficult. So I will switch to more mathematical ground.

A course is defined by a curriculum, so I would like to talk for a few minutes about the idea of curriculum. You may think you know what a curriculum is, but let me tell you what the word means to me

A curriculum is composed of four elements arranged in an expanding hierarchy. At the centre is the TEACHER, the most essential component. Just as education is fundamentally personal, so no part of the curriculum can be specified until we have been given the teacher.

Radiating out from the teacher is an array of PROBLEMS which the teacher thinks are important or interesting. The simplest and most direct way to describe the curriculum is to specify these problems. They are in fact the teacher's gift to the student.

A serious and sustained assault on these problems will lead the student to certain important IDEAS These ideas cover the problems like a mantle and in a sense define and give coherence to the course. But they are harder to describe than the problems and often remain implicit. In the hierarchy they stand outside the problems because they proceed from them, now as in the historical past. The ideas have become an important part of the discipline becuse of the problems that produced them

The ideas in turn give rise to certain TECHNIQUES These technical results are important because they help us to harness ideas generated from some problems to the solution of others. Certain technical skills are so useful they deserve to be mastered.

This then is a curriculum TEACHER, PROBLEMS, IDEAS, TECHNIQUES. The ordering is crucial Each level gains its significance and its authenticity from the level within My impression is that in the early schools, before the technological revolution, curricula were quite faithful to this model

Now let me tell you what the modern idea of curriculum has done to this superb model. Perhaps you have already guessed They have turned this organic hierarchy inside out Today's curricula are forced and unnatural They lack integrity and purpose. They have reversed a natural order The teacher who belongs at the center, is now on the periphery looking helplessly in, at a foul-smelling nucleus of miscellaneous techniques

Indeed, the central and definitive objects of the modern curriculum are the technical skills. They are presented to the student and illustrated with exercises (which bear no resemblance to what I have called problems) A weak attempt is made to derive ideas from the techniques but it doesn't come off because it's necessarily contrived The problems tend not to appear at all, and when they do they are certainly peripheral to the ideas and suggested only for the more able students Of course the students have not time anyway for problems since they spend all the time that they have any inclination to give on the exercises. And the teacher has no time to create any problems because he's too busy preparing lessons and marking exercises. And he certainly plays no fundamental part in the design of the curriculum. He is simply the waiter who sets the table and serves the food.

Let me give you a few examples, so you know what I mean by a problem, an idea and a technique.

1. Consider the problem of finding the infinite sum

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \ldots$$

In exploring this problem we look for simpler examples of the same type (geometric) we *can* sum We find $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, which clearly has sum 2 The reason we can see this is because each new partial sum cuts the distance between the previous partial sum and 2 in half. Does the original series employ a similar "distance-cutting" idea? We discover indeed that it cuts the distance to $\frac{3}{2}$ by a factor of $\frac{2}{3}$ each time. In attempting to generalize the idea we discover that every geometric series can be summed with this idea The technical device that emerges from the end is the formula $a/(1 - r)$

2. Now for a problem in sensitivity analysis. Consider the polynomial

$$P_\alpha(x) = x^3 + \alpha x^2 - 2x - 1$$

with parameter α. For $\alpha = 2$, the largest root of the polynomial is $x = 1$. How does this largest root change if α is moved slowly away from 2?

We use our calculator (hopefully programmable) to calculate the largest root x for values of $\alpha = 2 + u$ for u close to zero If we write x as $1 + v$ then we discover that the relationship between u and v appears to be linear (proportional) at least for small u, with v one fifth the size of u and opposite in sign. We look for a direct algebraic verification of this relationship. After a few such examples we discover the idea that all reasonable functions are locally linear, and hence a local sensitivity analysis requires the specification of a single number, the proportionality constant. We consider the new problem of how to find this constant for different functions at different points. The technique which emerges is the algorithm for calculating the derivative of a certain class of functions.

The two problems I have presented so far are rather special in that a particular and reasonably familiar idea emerges from each and a standard technique comes in turn from the idea. Thus these problems could fit into our modern curricula, although the practice is to present such problems *after* the technique has been covered, which seems to me to rob them of their significance But these problems are atypical, and you will not see many of them in this course The best problems usually suggest a number of ideas and techniques of different sorts. They do not fit into current curricula, and if imposed would be regarded as unsuitable They are of no use to teachers because they do not illustrate a particular technique Even teachers who enjoy them cannot justify spending the time they require.

3 My friend uses the following method of solving equations of the form $x = f(x)$ He programs his calculator to compute values of f. He starts with a value x_0 which he guesses is near a root and calculates $x_1 = f(x_0)$. He then reruns the program and calculates $x_2 = f(x_1)$. Then $x_3 = f(x_2)$. He continues this way until x_n and x_{n+1} are the same (on the calculator display) He then claims to have found an approximate solution Explore this technique. [This problem, like most others, requires the periodic guidance of the teacher, who generates good questions for the student to use as probes. Will the algorithm always terminate? How quickly? (how good is it?) What is happening geometrically? Try some particular examples [$\sin 2x$, $e^{x/3}$] Can you think of a better algorithm for locating roots?]

4. Let K be the set of all 3×3 matrices with entries ≥ 0 and all row and column sums equal to 1. We can regard K as a 4-dimensional polygon embedded in 9-dimensional space Describe K geometrically That is, visualize it.

Five problems like this would make an excellent grade 11 curriculum Two months on each Sadly enough, most of you have never had the experience of tackling such a problem, of chipping away at it piece by piece over a period of weeks, of choosing analogous problems that are simpler and perceiving patterns in their solutions which might generalize. The quiet thrill which builds up at the end is indescribable

The problem-solving process is the essential mathematical activity. It alone is the source of both motivation and understanding. So you may well ask, in dismay, why you have had so little of it in thirteen years of school That is an interesting question, and worth more time than I wish to spend on it right now. Let me simply suggest that there are two reasons: one is a perceived need by society for dependable technical skills, and the other is a perceived need by the educational system to standardize and measure its products.

These two reasons are not unrelated and have their origins only a century or two ago

The second of these reasons, the need to standardize and measure, I have little sympathy for. The first reason, the need for technical skill is worth consideration. For better or for worse we live in a highly technological society. The trouble is, if you try to impress technical skill with a stamp, you create a race of slaves, subservient to the technology which rules them. Like insects whose genetic stamp forces them into a parasitic relationship with a host, they are slaves because they are equipped to live in no other way. It is a fine life for an insect, but not for a man.

But it is a reasonable description of most of you right now. Canada is a nation of technological parasites

Ah well, you say, even if I am a parasite, at least I'm a competent one. I may not have many ideas, but I sure as hell have those technical skills.

It grieves me to inform you that you do not even have that comfort. As the year progresses you will be dismayed to discover how poor are even your simple manipulative skills. The genetic blueprint which instructs the insect so perfectly is a fine and sophisticated instrument beside the clumsy technological stamp wielded by our educators and textbook authors. The price of turning the curriculum inside out is paid at every level.

There, I have made you all quite gloomy. Little wonder. Who would not be gloomy at the discovery of thirteen wasted years! Sucked in by a system that promised so much and delivered so little. I can offer you one consolation which may help a little. Simply that it was not your fault. No one can really rest the blame on your shoulders. You were too young and naive to know any better. In those long interminable days when you sat in classroom after classroom bored to tears and pleaded under your breath, *God, why do I have to put up with this crap?*, it never really occurred to you that you didn't, that in fact there was an alternative, that, in fact, the whole thing was absurd and unnatural. You didn't have that confidence in the soundness of your own perceptions that would allow you to stand up in class and tell the teacher to *stuff it*. Having never seen it done right, you had nothing constructive to offer.

But now you have come to university. Am I to tell you that things will be different here? No I am not. You will not find the curriculum to be much improved. But there *is* a difference. Now you are no longer so young or so naive. Up to this point you could blame the system that has left its dreary mark on you. From this moment on, you have only yourselves to blame.

Let me warn you that if you take my course you will have some uncomfortable moments. You will not be allowed to take notes in class and that will make you uncomfortable. You will have to stand up before your classmates and produce a strategy for attacking the problem of the hour, and that will make you uncomfortable. You will not know how to study for my exams, and that will make you uncomfortable. And in November, a girl in another calculus class will rhyme off twelve topics that they have covered and ask you what you have done and you won't know what to say, and that will make you uncomfortable

But before the year is out you will be reciting poetry to that girl in a language, centuries old, that has inspired the world's greatest orators.

Postscript

I must confess that I have never actually given this lecture to my Calculus class, simply because I do not yet have enough material (problems) to stand behind these brave words. But I am working on that (and have been for three years), and I expect to have what I need quite soon. Perhaps fall 1980.

I have just read Peter Lancaster's excellent article in the CMS Notes 12, No. 6 (March 1980) on the transition from high school to university. He argues that mathematics should be taught more in a cultural context (more applications-oriented problems), an emphasis I support, and quotes many of my heroes (Whitehead, Morris Kline, Polya, and of course, Coleman, Higginson and Wheeler, but not, alas, Virgil). Let me quote him quoting Kline quoting Schiffer.

> *The miracle of mathematics is that paper work can be related to the world we live in. With pen or pencil we can hitch a pair of scales to a star and weigh the moon. Such possibilities give applied mathematics its vital fascination. Can any subject give the would-be mathematician — initially at least — a stronger and more natural interest?*

I agree, but I must issue a strong caveat. There is only one way to create a strong and lasting interest in mathematics and that is to make the subject (material) belong to the student. We are interested in that which is ours. This is the ultimate and the *only truly natural* source of motivation

Too often I have seen an instructor present a truly marvelous application, whose compelling nature is designed to coax even the most recalcitrant student to struggle with the analysis. Alas, only the clever student is effectively challenged. The others see the real life situation to be something they ought to be interested in, but something that necessarily makes it much harder to figure out *what I'm supposed to do*. The effect is that in the end they learn what they think they're supposed to do, but record one more piece of evidence that, in the hands of ordinary people like themselves, mathematics is not a realistic tool for coming to terms with the outside world.

In fact, the outside world can be just as good a source of the sort of exploratory problems I am advocating as numbers, curves, equations and triangles. Indeed Lancaster includes a number of such problems at the end of his article. But in all cases, the crucial requirement for engaging the student is that he have substantial creative input in the exploratory process. And this process must be at the heart of the curriculum

And this process must be at the heart of the curriculum. I feel I must elaborate on this last sentence for those who are simply reading this essay and are not students in Math 120 (who will find out what I mean soon enough). It does not mean a shift in emphasis away from the standard curriculum, perhaps the inclusion of a few more challenging problems. Rather I intend a fundamental reorientation: truly a turning inside out.

The teacher is at the centre of the curriculum because it is those few hours that the student spends with the teacher which determine for the most part the quality of his or her interaction with the subject. The nature of the classroom activity has a definitive impact on the way time is spent outside the classroom. This is the reason it is so destructive to lay out technique in class or present the solution to a problem the student has not even had a chance to think about.

If we want mathematics to provoke, excite, and challenge the student then the teacher must provoke, excite and challenge him in the classroom. If we want the student to engage in the mathematical process then he must do so in the classroom.

Here then is my idea of classroom activity. At the beginning, the teacher presents a problem. It must be accessible to the students, in that they must be able to understand the problem and begin to explore it. The hour is spent in a group exploration (no more than 30 per class please!), with the teacher acting as a subtle guide and consolidator. In the next class the problem may continue or a new one appear. In between classes the student comes to terms with what has happened and considers variations. Periodically several problems will come together to suggest some unifying ideas. These can be identified and discussed. Techniques will also be identified based on the manipulations required by the problems. Those that are judged to be important must be practised. The students should be encouraged to create "exercises" for this purpose.

The key is the problems. Given those, with the guidance through them provided by the teacher, the students can do the rest themselves. And should.

ACKNOWLEDGMENT

The Editor and Publishers are pleased to acknowledge the assistance they have received from Concordia University, Montreal in the publication of the first issues of this journal

Contributors

J. AGASSI
Department of Philosophy, Boston University
745 Commonwealth Avenue, Boston, MA 02215, U.S.A.

J. A. EASLEY, Jr.
Committee on Culture and Cognition
University of Illinois at Urbana-Champaign
130 Education Building, Urbana, IL 61801, U.S.A.

C. GATTEGNO
Educational Solutions Inc
80 Fifth Avenue, New York, NY 10011, U.S.A.

P. NESHER
Mathematics Education Division
School of Education, Haifa University
Mount Carmel, Haifa, Israel

H. RADATZ
Universität Göttingen, Fachbereich Erziehungswissenschaften
Waldweg 26, 3400 Göttingen, West Germany

D. G. TAHTA
Cold Harbour, Newton St Cyres, Exeter, Devon, England

P. D. TAYLOR
Department of Mathematics and Statistics, Queen's University
Kingston, Ontario, Canada K7L 3N6

J.V. TRIVETT
Faculty of Education, Simon Fraser University
Burnaby, British Columbia, Canada V5A 1S6