

## **The tyranny of reality.**

*They started from opposite ends. Pisano is a master of form and is striving towards reality; Julia aches to throw off the tyranny of reality and reach the essential that lies somewhere underneath.*  
The Dream of Scipio, Iain Pears p. 279.

*It is a powerful weapon, yet it's aesthetically superb.*

Tom Cruise in awe of the art that has gone into the making of his samurai sword. Kingston Whig Standard, Sept. 5, 2003, p.28.

My premise is that we mathematicians are sitting on a gold mine. In terms of structural beauty, stunning insights, unexpected power, all from simple, accessible, ingredients, very little can compare with our wonderful subject. But we do a terrible job at communicating that to most of our students. In the classroom we blow it, and we thereby alienate just about the entire population. And we've no one to blame but ourselves.

You've heard this before, possibly even said it. Lots of compelling essays have appeared, laced with phrases such as "exploratory problem solving," "engaging the student," "less is more." But change has been glacial at best. Maybe at age 60 I'm becoming impatient, but I am too aware that here again is a generation of students that we are losing, and who are losing us(!). And I'm counting not only the loss in math majors, though that's significant enough, but a widespread loss of allegiance, of feeling at home with us. Because of the scientific and economic importance of our subject (and if it wasn't for that we'd be in deep trouble) we get more students at the introductory level than any other discipline. I want most of them to come away from their math course with a sense of belonging, with a sense of a new domain that they are a part of rather than apart from.

These issues are discussed in Math Departments and at professional meetings. It is suggested that we need to teach differently, but perhaps that's not the best way to say it. Most of us are pretty good teachers already, but if the process is to have integrity a teacher is only as good as the material being taught. And what we need is a new curriculum, for example new kinds of problems that are imaginative and engaging and work well in a large lecture theatre.

Well hang on here. Surely a lot of that has been done over the past 20 years (reform calculus, for example) and indeed it remains today an active field of development. Yes, that's true. So why haven't we seen more of an impact?

The point is that for years we've talked about this and we've talked about this and we've talked about this, and almost nothing has happened.

I believe the problem is that there are *two* things we have to do. One of these is to find a new way to teach and that's what most of our reform efforts have been working on. [Indeed for me this is mostly about curriculum, at least I consider teaching methodology as being based in or driven by curriculum.] But the other is to let go of the old way, and I have a feeling that's a lot more difficult, or at least it poses a much more subtle problem.

I see this when I share with colleagues some neat exploratory problems that might work well in their courses. They unflinchingly like the problems, but in their wrinkled foreheads I can see a calculation of the time it will take. "What can I afford to leave out?" That's "old way" thinking, and if we don't let go of that, we will never successfully embrace the new.

Letting go is hard. To succeed I believe that we need a different model about what it is that we are doing. I believe that we need a new metaphor.

*Picasso's Guernica 1937.*



In the afternoon of April 26, 1937, German bombers, flying for Franco, annihilated the defenseless Spanish town of Guernica, the centre of the Basque cultural tradition. For over three hours, a powerful fleet of bombers and fighters circled and wheeled over the town, dropping thousands of bombs, and setting everything on fire. The fighters then dropped low to spatter with machine gun fire those who had fled to the fields.

Over the next few days, the news of the massacre at Guernica spread to a shocked and outraged world. It was not the first of Franco's atrocities, but it was the one which galvanized Picasso into action. He had already accepted a commission for a mural at the Spanish pavilion at the Paris World fair, but he had so far produced nothing. In the six weeks following Guernica, he worked at a feverish pitch to produce a memorial to the innocent dead and a manifesto against the brutality of modern war.

The painting is 26 feet wide and 11 feet high. The figures rage across the canvas in a rush of terror. Heads everywhere are flung high, mouths forced open in a frozen outcry. A jagged light casts its sharp illumination on the scene. A woman from the outside world leans through the window surveying the carnage with a feeble lamp, her face a mask of horror. Except for the harsh whites, everything is dark, claustrophobic, compressed in gloom. The images are stark and simple, almost childlike, a woman and a child, a peasant woman, farm animals, a single stricken household says it all. [Excerpted in part from Life 65, December 1968 pp. 86-93.]

*The way of the artist.* A work of art is a representation of reality, a representation subject to certain essential constraints (the canvas, the sonnet, the steps of the dance). However the *objective* of the work is not in fact to *represent* but to *transform*, to transform our perception of the reality, to allow us to see what's truly there, to open our eyes, to free and empower us. It accomplishes this by stripping away the inessential aspects of the experience, and rendering with imagination the simple lines that remain. This imaginative transformation is such that the work, if successful, *conveys the experience more sharply and truly than can reality itself*. In this way, art, which, because of its self-appointed constraints of form and structure appears to work at a disadvantage, manages to turn these constraints into a more focused, more memorable, more telling experience

than the real thing. That word “telling” is a good one here because the raw experience itself is often overlaid with complexities and irrelevancies which interfere with our attention. Art, as a highly particular retelling, focuses us and allows us to listen in a new way.

An interesting example we are perhaps all familiar with is the movie *A Beautiful Mind* that attempts to provide an artistic portrayal (within a certain medium, that being the genre of big Hollywood films) of the life of the John Nash. This is all the more interesting because, though it is widely regarded as having succeeded on a number of levels as a work of art, it was criticized for departing significantly from Nash’s life. But the important point (well made by Keith Devlin and others) is that the movie is not a “photograph” of the life lived. If it had attempted to be that it would almost certainly not have worked in that particular artistic context. Instead it took on the (formidable) challenge of capturing the essence of that life (both personally and mathematically) in a 3-hour Hollywood-style film, and by most accounts succeeded wonderfully. For those who want more (and the movie has almost certainly inspired many to seek out more) there are always books and webs, for example, Sylvia Nasar’s excellent book of the same name.

Time for a shift in mood; here is another Picasso. Completely different from *Guernica*, *Don Quixote* is simple, gentle, patient, whimsical, and memorable.



As teachers of mathematics we are artists. The landscape we gaze upon, brush in hand, is a coherent body of mathematical ideas and results. It is however not our job to thrust this body of results upon our students. Rather our challenge as artists is first to “strip away the inessential aspects,” and then to render imaginatively “the simple lines that remain.” This stripping away is quite different from asking, “what can I leave out?” If you ask the artist what she has left out of her picture, she might regard you with puzzled amusement, and then reply, “Everything; I pitched the lot,” but she might just as well reply, “Nothing; everything is there.” Indeed, just as art is less than reality, so the problems and explorations we conjure up will be less than the whole mathematical theory. And just as art is so much more than reality, sharper, more focused, more particular, *so these problems can convey the true mathematical experience better than could the mathematics itself.*

Restraint is a key component of artistic integrity and here it comes down to trusting the problems to do the work they are designed to do. That's the "letting go" part and it's not easy—caught up in the complexities of the subject, we are too forcefully aware of so much that has to be said, explained, clarified, and we are seized with doubts that the few students who actually might need something that we have left out will be able to capture that on their own. But the rewards of restraint can be enormous. It gives room for the encounter to continue to work (and play!) in the mathematical lives of our students, and it encourages them to be artistic in their own efforts.

An example might help, and I choose one from my introductory linear algebra course. A central concept in the course is the notion of eigenvector, or more generally of *eigensolution* that being a special solution which has the virtue of being easy to describe, but has the vice of not being a solution to the problem at hand. But it *is* the solution to a closely related one and the idea is that with luck (and linearity) we can put these special solutions together to get the solution we are after. This strategy is so central to the subject, that I build a large canvas around it, large enough to occupy an entire third of the two-semester course. I begin by *counting trains*. This is a simple exploratory problem with lots of fine side-roads (for example massive explorations into Fibonacci numbers), which contains the essence of the idea of eigenfunction expansion.

*Problem 1. Counting trains.* I am constructing trains using cars that are either 1 unit long or 2 units long, where there is one type of car of length 1 but two kinds of cars of length 2, type A and type B. Let  $t_n$  be the number of trains of total length  $n$ . For example  $t_3 = 5$ , the 5 different 3-trains being 111, 1A, A1, 1B, B1. [Note that trains are ordered so that 1A and A1 are indeed different.] Find a formula for  $t_n$  in terms of  $n$ .

*Solution.* By counting, students can generate a number of terms of the sequence: 1, 3, 5, 11, 21... I suggest the possibility of recursive thinking and they eventually come up with the argument that

$$t_n = t_{n-1} + 2t_{n-2}.$$

[Count the number of  $n$ -trains conditional on the first car.] This leads to the initial value problem:

$$t_n = t_{n-1} + 2t_{n-2} \quad t_0 = 1, \quad t_1 = 1.$$

Armed with this, the students can easily generate more terms:

$$1, 3, 5, 11, 21, 43, 85, 171 \dots$$

Many students see that each term is twice the preceding term except you alternately add or subtract 1. By comparing terms with powers of 2, they are led to the formula:

$$t_n = \frac{2}{3}2^n + \frac{1}{3}(-1)^n.$$

It's a nice formula and it fits the terms so far, but can we be sure it will work forever? [One way to prove this is with mathematical induction, but I'm after a bigger game here. We will look at induction later in the course.]

Here's where I put forward our fundamental strategy: *look for alternative initial conditions that have simple solutions. Then try to use these as building blocks to construct other solutions.* What the students find (perhaps by trial and error, trying different initial conditions) are the geometric sequences  $\{2^n\}$  and  $\{(-1)^n\}$ . And then we argue that sums and scalar multiples of solutions are solutions and we manage to write the solution we are after as a linear combination of the geometric solutions, and we have found a rigorous argument for our formula.

This is the problem that introduces the general notion of eigenvector. From here we go on to study a number of standard matrix recursions (age-structured population growth, systems of brine tanks, equilibrium price vectors, etc.) In each of the past two years I have restricted myself to real eigenvalues, partly because I wanted to do justice to the above (real) examples, but also because the last time I “did” complex eigenvalues, the students found it difficult and it did seem to take a long time. But again this year the question arose. Can I include complex eigenvalues? What would I have to leave out?

And I suddenly realize I’ve fallen into the same “old ways” of thinking that I have warned others to avoid. I have been automatically assuming that to “do” complex numbers would entail a whole bag of stuff—complex arithmetic, trigonometry, and enough examples of different kinds to “cover all the angles.” But why not just do one example—a well chosen work of art that convey the magic of the topic, shows off the power of our brave decision to try to push through with complex eigenvalues an idea that we previously realized with real ones. For example:

*Problem 2.* Solve the recursive equation  $t_{n+1} = 2t_n - 2t_{n-1}$   $t_0 = 1, t_1 = 3$ .

*Solution.* If we tabulate the first 12 values

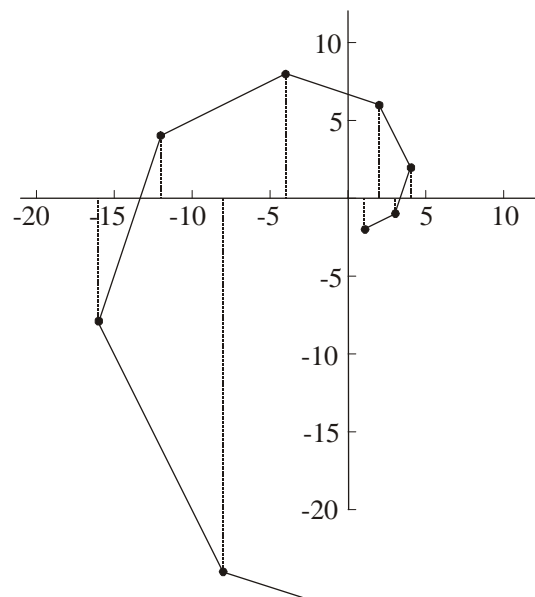
$$1, 3, 4, 2, -4, -12, -16, -8, 16, 48, 64, 32,$$

we perceive a block pattern with blocks of size 4. From here we could again use mathematical induction to show that the pattern continues, but we actually want to “see” how the pattern unfolds. The students are used to looking for “multipliers,” and here they find one in  $-4$  but the trouble is that it seems to take 4 terms to act. How might we encapsulate that? Could such a “jerky” pattern ever be described by any kind of natural construction?

Using the train technique, we look for geometric solutions  $\{r^n\}$  to the equation and we find two with  $r = 1 \pm i$ . Now these are complex, but we push forward in spite of that. [A precocious student might be unable to resist calculating  $(1+i)^4$  and getting  $-4$ . What a discovery!] We try to write our target sequence as a linear combination of the two geometric sequences and since the two terms of the sum are conjugates, we get a sum of conjugates which can be written as the real part of a sequence of complex numbers. We get:

$$t_n = \operatorname{Re}[(1-2i)(1+i)^n].$$

The sequence in the square brackets is geometric (with multiplier  $1+i$ ), and it is therefore a spiral in the complex plane with a  $45^\circ$  rotation each term. The projection of this on the real axis is our desired sequence. This is a lovely example of the visual power of embracing the imaginary dimension—the spiral is seen as a *deus ex machina* that generates the sequence from above, as it were. And in displaying multiplication as rotation it showcases the fundamental contribution that complex numbers make to our understanding of arithmetic.



What of the rest of the course? How does it develop? Which ideas, which theorems, which technical results? For example, do I go farther with complex eigenvectors? Do I go on to a  $2 \times 2$  matrix equation where I use the same complex plane representation to track both  $x$  and  $y$  together? Should I get into change of basis stuff (something I've actually so far done without, even with the real eigenvalues)?

Such questions as these we always struggle with, and they are very much the struggle between Pisano and Julia. In this process we are guided in our thinking and feeling the way an artist is so guided. The course evolves as does a painting grows or a dramatic work. We draw on our deep knowledge of the landscape, on the character of the work and the nature of the artistic medium. In this we must look clearly and carefully; we must strive to "see" with fresh eyes. As the work grows, so do the possibilities. But there's an essential closing down as well. Each new piece must fit the emerging whole. It's a question of integrity. [What distressed me about our recent high school curriculum revisions was a blatant disregard of this principle. Topics were stuffed in here and there with little connection to the whole.]

In an article a few years ago, William Kirwan, mathematician and President of Ohio State University, called for "a reshaping and restructuring of the curriculum with greater emphasis on active learning at all levels." The ideas put forward here are exactly that—a reshaping and restructuring. It is however a big change. It questions the very canon of the subject, at least at the introductory level. To do it right requires many creative ideas—and courage as well.

William Kirwan, *Mathematics departments in the 21<sup>st</sup> century: role, relevance and responsibility* MAA Monthly 108, January 2001.

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