MATHEMATICAL lens

Peter Taylor

Arches, King Fahd University of Petroleum and Minerals, Dharan, Saudi Arabia

In September 2006, I was at the King Fahd University of Petroleum and Minerals in Saudi Arabia for a workshop on preparing high school students for university. Though Saudi Arabian culture is very different from ours, Saudis have the same problems in getting their students excited about the study of mathematics, and indeed that was the focus of the workshop. The photograph shows me standing inside a particularly fine walkway together with (from left) Hussain Al-Attas, the director of first-year studies, and Suliman Al-Homidan, the head of the mathematics department. Thinking there must be a good mathematics problem somewhere, I asked my Australian colleague Peter Galbraith to take the photograph.

And indeed there is! Recently, I gave the problem to a group of grade 11 and 12 students, and what happened was quite fascinating. Here's the problem.

PROBLEM

The arches that recede into the background of the photograph are equally spaced and are the same size. But in the photograph, their size seems to decrease the farther back they are. The question is, exactly how do they decrease?

(*a*) Take the measure of "size" to be the width of the horizontal opening of the arch, as measured with a ruler on the photograph. Let w_i be the width of the ith arch on the page (use the front face of the arch). State the form of this dependence—that is, how w_i depends on i. Your expression for w_i will have some parameters, as there are physical measurements you are not given. So, what is really wanted here is the *functional form* of the dependence.

(*b*) Suppose you are told that the arches are spaced at intervals of 2.5 meters. How far away is the camera from the front face of the first arch?

"Mathematical Lens" uses photographs as a springboard for mathematical inquiry. The goal of this department is to encourage readers to see patterns and relationships that they can think about and extend in a mathematically playful way.

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DISCUSSION AND SOLUTION

(*a*) Most students hardly knew where to begin, as there was nothing here they recognized as a mathematics problem with a known method of solution. Much can be learned from their attempts at a solution, and I will begin with these.

Finding the best curve

The idea of actually gathering some data (making measurements) caught on, and soon everyone had a list of widths for at least 6 arches. I had an Excel sheet projected at the front of the class, and the data was plotted for all to see (see **fig. 1**). What happened next was (to me) completely unexpected. The students started vying to see who could produce the function that gave the best fit. Actually, that should not have surprised me at all. They are, after all, exactly what we have made them to be—true children of technology. Let's pursue that story.

What kind of curve does that data set evoke? Believe it or not, some students tried polynomials. A cubic polynomial gives an impressive $R^2 = 0.9985$ (see **fig. 2**), but a quartic gives $R^2 = 1$ (see **fig. 3**), and one cannot do any better than that. The student in question deduced that the right answer must be the quartic polynomial shown in **figure 3**. (One thing that should be emphasized at some point, though perhaps not at this juncture, is that this is definitely not what R^2 is supposed to be about.) While this quartic model fits the given data well, it fails to provide a realistic answer for values of $i > 6$.

Two other popular choices were the exponential and the power function. The exponential form, $w_i = ar^i$, is clearly a poor fit (see **fig. 4**), but some of its proponents remained stubbornly committed to it nonetheless. One group calculated a number of the ratios $r = w_1/w_2$, w_2/w_3, w_3/w_4, etc., and actually concluded from this that the ratios were *not* constant, which earned them my approval. The power function looks pretty good (see **fig. 5**), and it is, in fact, close in spirit to the correct answer.

At this point, the class took a vote and chose the power function. It does not give the perfect fit of the polynomial, but it is simpler, and I guess they thought it had a better chance of being right.

What was I to do? I had a class of curve fitters on my hands. How was I to get them to imagine that there was a totally different way to tackle the problem?

What I did was to put the problem shown in **figure 6** on the board. Consider the following nested sequence of squares. Find a formula for the side-length s_i of the ith square. Suppose the size of the largest square is $s_1 = 4$ cm. What is the size of the ith square?

Would a solution depend on measuring the sequence of side lengths and finding a curve that gave a good fit?

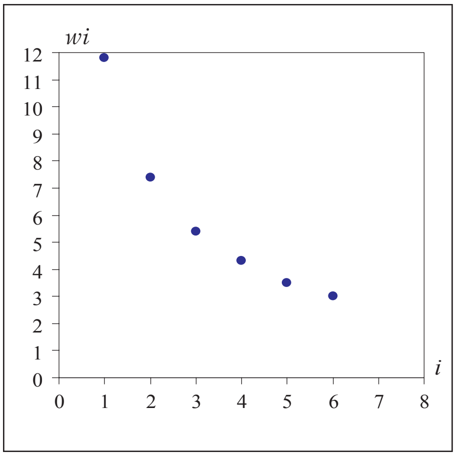

Fig. 1 Spreadsheet data plot. Note that students were working with a full-page photograph.

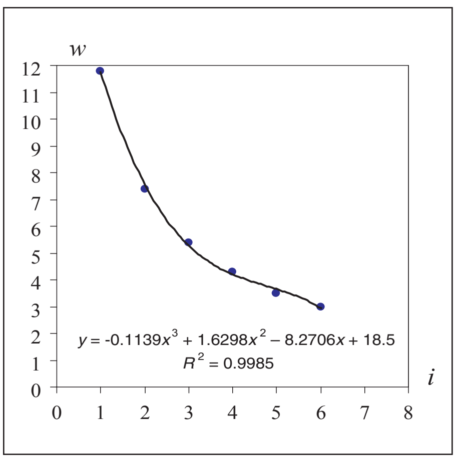

Fig. 2 A cubic fit

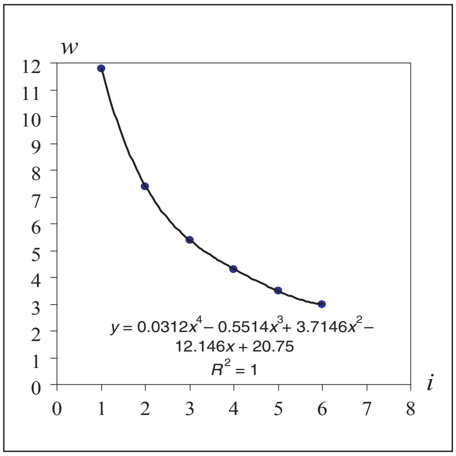

Fig. 3 A quartic fit

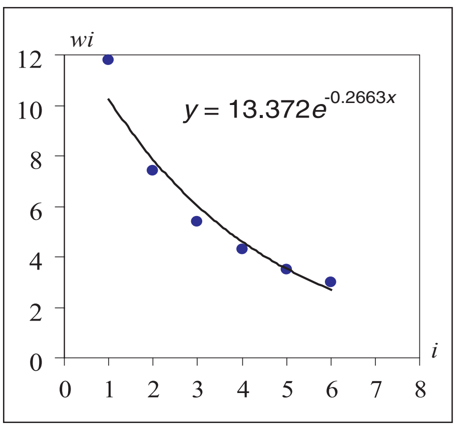

Fig. 4 Exponential fit

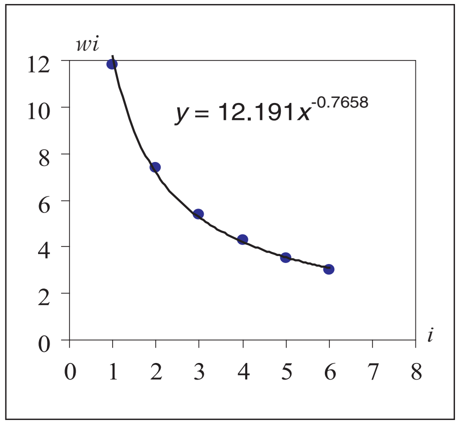

Fig. 5 Power function fit

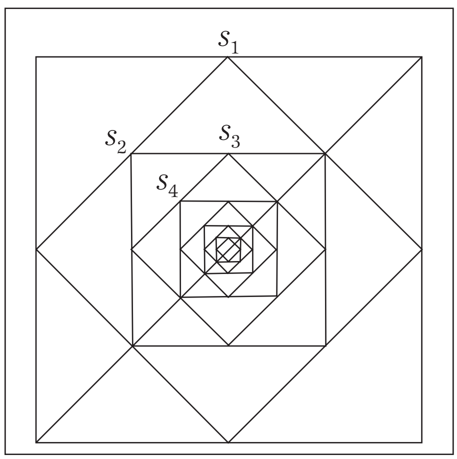

Fig. 6 Nested sequence of squares

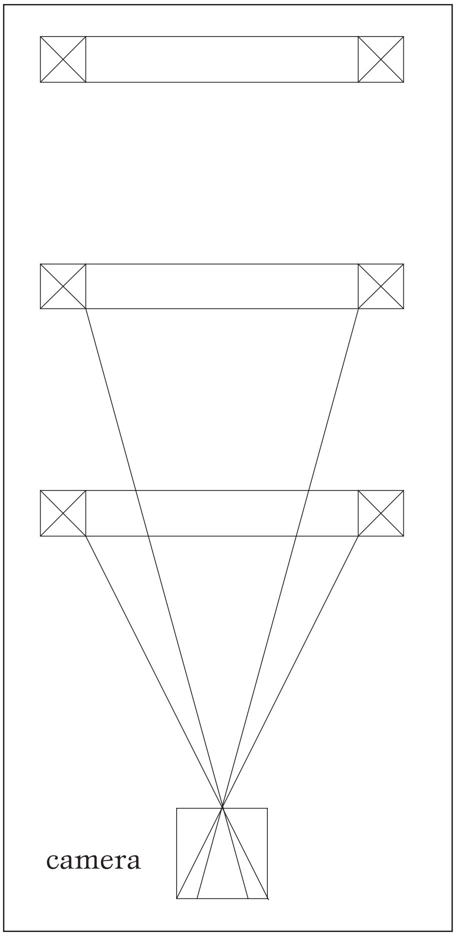

Fig. 7 A useful diagram includes the camera as well as the arches

My students got the point. They started to do some geometry.

When students were asked if the arches problem were something like that, we started a conversation about cameras and light rays.

The geometers

After considering the problem, I concluded that the diagram in **figure 7** is the one that needs to be drawn, because it includes the camera and its components, like the aperture and the film. Some students, who represented the camera as a point, were prevented immediately from thinking clearly about what was really happening, and so it was hard to make any sense of their "arguments."

When the diagram is made into a schematic and variables are introduced, as shown in **figure 8**, similar triangles immediately give us the form of the equation relating the width of the *i*th arch in the picture, w_i, to the real width, w, of the arches. For example, the second arch gives us

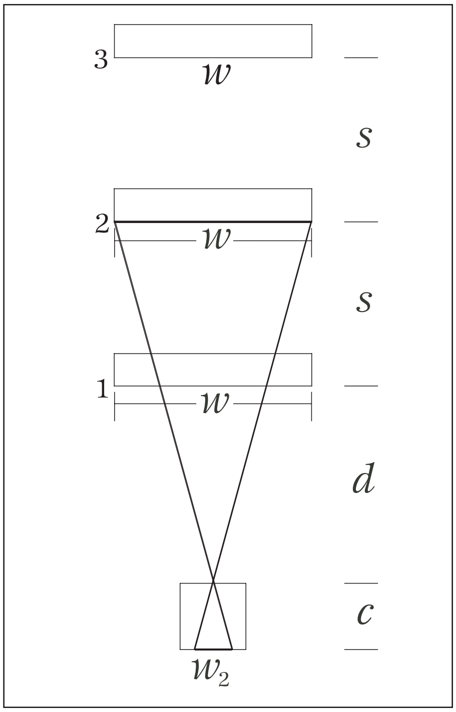

Fig. 8 Variables are introduced to make the similar triangles more obvious.

$$\frac{w_2}{c} = \frac{w}{d+s}.$$

Here, c is the depth of the camera, d is the distance of the camera from the first arch, and s is the spacing between arches. The corresponding formula for the ith arch $(i \geq 1)$ is

$$\frac{w_i}{c} = \frac{w}{d+(i-1)s}$$

or

$$w_i = \frac{cw}{d-s+is}. \qquad (1)$$

This provides the form of the dependence of w_i on i. More abstractly, we could say that it has the form

$$w_i = \frac{1}{a+bi}.$$

A few students worked with a diagram like the one in **figure 9**, thinking of the image as being captured on a "screen" placed in front of the eye. There is no camera here, but they are thinking of the image as if the viewer looked at the scene through a window with the image etched on a pane of glass. This solution got full marks.

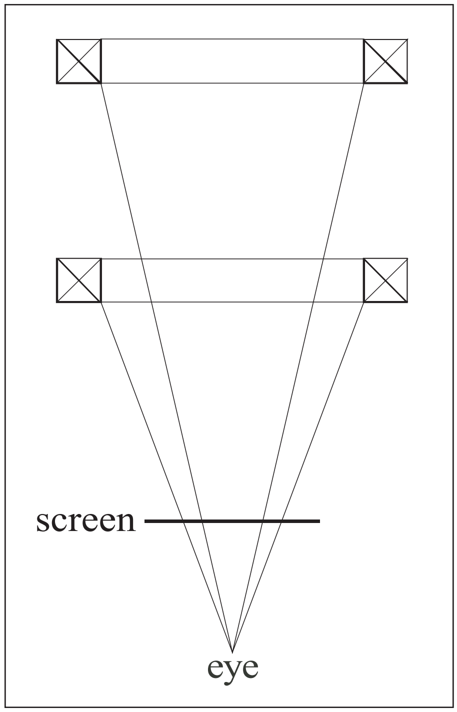

Fig. 9 An alternate solution employed by students

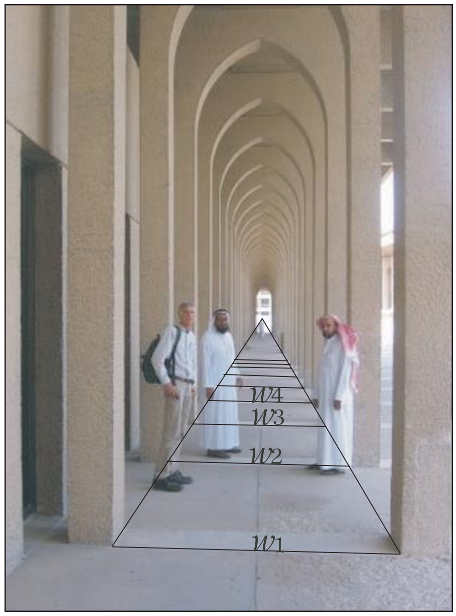

Fig. 10 Despite the many similar triangles, students who used this to try to determine a solution fell short.

A number of students worked with the diagram they simply drew on the picture shown in **figure 10**. There are many similar triangles here and many chances to work with angles and use results from trigonometry, and it was a bit heartbreaking to see them struggle away, saying things about perspective and the significance of the point at infinity, and writing

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down everything they thought might be relevant but without any hope of getting a coherent solution.

What to conclude from this? That from an early age our students need more experience in doing what mathematicians do best: shining a careful, precise mathematical light into a corner of the world and coming up with an explanation of what is really happening and, perhaps, why.

(b) Here we are told that the arches are spaced at intervals of 2.5 meters and are asked for the distance from the camera to the front face of the first arch. We are given $s = 250$ (cm), and we are asked to find d (see **fig. 8**), but cw is unknown. However, taking the quotient of two successive w_i from equation (1) will eliminate it. Most students did this and chose w_1 and w_2:

$$\frac{w_1}{w_2} = \frac{d+s}{d}$$
$$dw_1 = (d+s)w_2$$
$$dw_1 = dw_2 + sw_2$$
$$dw_1 - dw_2 = sw_2$$
$$d(w_1 - w_2) = sw_2$$
$$d = \frac{sw_2}{w_1 - w_2}$$

Using the values from part (a), working in cm:

$$d = \frac{250 \times 7.4}{11.8 - 7.4} \approx 420$$

The camera is 4.2 meters from the first arch.

Those students who used the "screen" diagram got the right answer if they calculated the distance from the first arch to the eye.

Of course, I gave this solution full marks. But I am partial to an approach that uses all the measurements that were made. As a bonus, it can give us a verification of our model.

Finding a straight line

Whenever I am fitting a curve to data (see **fig. 1**), I like to transform the data, if possible, so that the transformed graph is a straight line. The reason for doing this is twofold. First, straight lines are easy to "see" and can therefore provide a simple check on the model. In addition, the equation of the trend line, in using all the data, can give us best-fit values of our parameters.

In this case, we have a functional relation between w_i and i of the form

$$w_i = \frac{cw}{d - s + is},$$

which is (1).

We can obtain a form that is linear in i by working with $1/w_i$:

$$\frac{1}{w_i} = \frac{d-s}{cw} + \frac{s}{cw}i$$

This tells us that a plot of $1/w_i$ against i should be a straight line (see **fig. 11**). That is a very satisfying way to validate our geometric argument.

One thing we might do now is get best-fit values for our parameters from a trend line (see **fig. 12**). Technology gives us the line

$$\frac{1}{w_i} = 0.0352 + 0.0498i.$$

Thus,

$$\frac{d-s}{cw} = 0.0352 \quad \text{and} \quad \frac{s}{cw} = 0.0498.$$

Taking the ratio of these two equations yields

$$\frac{d-s}{s} = \frac{352}{498}$$
$$498d = (498 + 352)s = 850s = 850 \times 250$$
$$d = \frac{850 \times 250}{498} \approx 427.$$

The answer we got before, using only w_1 and w_2, was $d \approx 420$. The difference between the two approaches is only 7 cm, but which one is apt to be better? That is an interesting question.

Which of the two answers is better: the first one, which used only w_1 and w_2; or the second one, which used all the data? One might argue for the second, as it uses more data; however, best-fit lines (which are now available at the touch of a student's fingertips) need to be used with care. One key assumption behind linear regression is that the size of the measurement error is the same for all data points, and this is not likely true for the values of $1/w_i$. If the w_i were all measured with the same accuracy, the error in $1/w_i$ would be much bigger for the arches that are farther away (can you see why this is the case?). Theory says that in the least-squares minimalization, these would need to be given less weight, and the standard regression line does not do that. So of our two solutions, the first one gives the first two arches too much weight, and the second give them too little weight. Maybe the real answer is somewhere in between. ∞

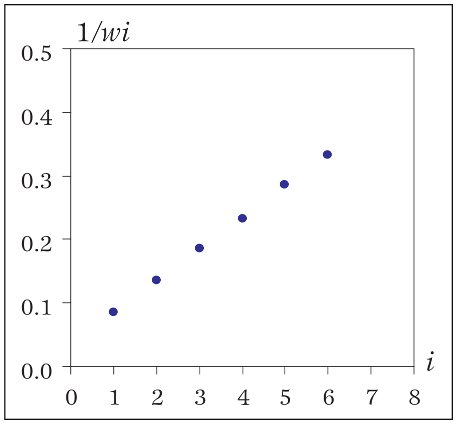


Fig. 11 Transformed data

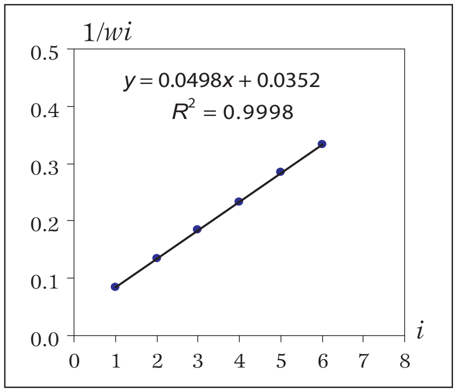
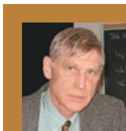


Fig. 12 Trend line on transformed data



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