Essay on A and B

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In the last class of my senior *Math Explorations* course, I asked students about their undergraduate mathematics experience. I was curious, because during past years of the course, I noticed patterns that were repeated by this year's class more strongly than ever. First, they were generally mathematically weak, unexpectedly so, given that they were third or fourth year mathematics majors (or joint majors). They seemed to "know" a lot, but could do little with it. Secondly, their weakness was especially pronounced in analysis, and they did not much enjoy problems in this area. (I will give an example at the end.) Thirdly, in the last three weeks, when groups of four chose their own problems to use in "working the class", they inevitably chose problems that might be found in recreational mathematics books.

Hey, there were a lot of positives. They loved being in mathematics, being part of the mathematics "culture"; they loved neat problems; they loved working together, solving problems together, presenting together. They were clever in lots of neat ways and were in fact a wonderful group to teach and have fun with.

From the discussion of that last class, I am sure I understand where much of their apparent weakness comes from. Briefly, in their first two years of honours mathematics, they have had too little time to "play". This was so in most of their mathematics courses, but it was particularly acute in the analysis sequence. They felt they never had the time to internalize the material, to throw the concepts into alternative forms and see what emerged, to be director of the show. They told me this with a good knowledge of what it means to "play" because that is what they have been doing all semester with the material. As you might expect from the title of the course, we work with problems from a range of areas: geometry, probability, calculus, algebra, number theory, logic — some of it even at the high school level, but always problems with a kick, a mystery, a challenge, an unexpected structural turn.

As a huge oversimplification, I suggest that there are two "streams" of students in our honours program and I will use the labels A and B to differentiate them. The A's are not hard to describe. They come to us with pretty good mathematical learning skills, and in particular a good sense of *pace*, of how long to listen before they know it is time to go off on their own. They are keen to explore concepts of proof and mathematical rigour, and they have some commitment to study mathematics further and more deeply, perhaps at the graduate level, perhaps in physics, bioinformatics, finance or neuroscience. They will be happy with a number of good lectures, but they do not need many of these. Mainly, they need a good set of problems, and some good faculty mentors. For the rest, they will learn by teaching one another. They are already quite independent. They comprise some 25% of our majors.

The remaining 75%, the B's are harder to characterize. They represent a large diversity of abilities, commitments, interests, destinations and needs. From their ranks will come many of our next cohort of leaders, and many of our most devoted alumni. They have a wonderful energy. They are the heart, though perhaps not the soul, of the Department. Though their needs are diverse, I will try to identify a few of them. They do not need much specialized knowledge, but they need to learn how to learn and how to gain mastery; they need to learn how to think clearly, read incisively, and write and speak simply, perhaps elegantly and passionately; most of them arrive thinking university to be a super high school, so they are ripe for a transformative experience in first year; in their senior years, they do not require a narrow, specialized, comprehensive treatment of any branch of mathematics; rather they need to develop their skills of research and communication.

My view is that the first two years of our honours mathematics courses work extremely well for the A's. My judgment is based on my knowledge of the curriculum, my knowledge of those of my colleagues who have been teaching these courses for the past years, and many conversations I have had with A students. For the future PhD, these courses are a superb initiation into the soul of mathematics.

But I am convinced that the first two years of our program do not work nearly so well for the B-students. This dichotomy emerges from many factors, but a significant one (and one of current interest to me) is the fact that in those critical two years, the A-students play with the material and the B-students do not. The A's play out of interest and the desire to learn well, but most of all, they play because they already know how to play. Somehow it has been part of their mathematical lives for quite a while and that is perhaps a significant reason they came to be A's. The B's did not play much in high school because they never had to. And they failed to pick up that particular skill in university (although they could see their professors at play) because, faced with more to learn that they felt they could handle, they found themselves too often short of time.

The students who enrol in my explorations course (which is one of the courses in our teaching focus) are almost exclusively B's. With one or two exceptions, they tell me that the teaching focus courses (of which we have three) provide the first occasion in university mathematics courses in which they really felt able to play. Analysis courses were cited as a particular barrier for them in this regard. That is not surprising. Calculus is a harder subject to "get your hands on" than is algebra or geometry.

So, where am I headed with this? I begin with an impractical suggestion.

First suggestion: Offer two versions of our honours courses in the first two years: an A-course and a B-course. The A-course would be pretty much what we offer now, except that the teacher and students would be free to use a more research-based format. The B-course would have less material and more emphasis on mastery, on how to learn through play. Take calculus as an example. It is important to emphasize that this B-course would *not* be similar to the standard service courses we all offer. It would be theoretical and conceptual because these students do love the structure and beauty of the discipline.

This first suggestions is not a solution at all for a university, such as mine, of a fairly small size, particularly, in times of financial constraint. And for other reasons I'll soon mention, it is a bad idea anyway. So I move to the second suggestion.

Second suggestion: Offer only the B-version of the courses described above.

Come again? What will the A-students do?

They'll take the B-courses too.

But I thought the A-courses were just right for them.

They are. They were.

So we're shortchanging them?

Maybe. But maybe not. Perhaps it would be the best thing that could happen to them.

The point is, as I said before, that it does not matter a whole lot *what* we teach the A-students. They have an internal agenda and as long as they have good problems and good mentors, they will accomplish it. In many ways, the calculus course I took (from John Coleman) in the early sixties was a B-type course. It did not have a lot of material (it used Ralph Jeffery's slender little book), and we were pretty well left on our own to prove the theorems and often even to formulate them precisely. John told us just enough about epsilon and delta to whet our appetites. Though I must confess that things were kinder in the early sixties; classes were smaller and there was less pressure, both time pressure (a simpler world with fewer distractions) and performance pressure (we did not have to care so much about marks).

There is a lot to be said for keeping the A's and B's together; they have a lot to offer one another.

This essay is perhaps a more particular or more transparent version of an old argument that we should put less material into our courses. Maybe the current cost-crunches make the time ripe for some ancient wisdom.

Example. *A game of competition.* Suppose that there is a task that requires two persons, but when carried out provides a benefit to each. However, the amount of effort that each partner invests in the task can vary, and therefore the individual cost can vary too. Finally, the benefit gained by each individual depends on the total investment.

Introduce some notation. Let x ($0 \leq x \leq 1$) denote the investment of a typical player and let y ($0 \leq y \leq 1$) denote the investment of the player's partner. The net payoff to the player will have the form

$$P(x; y) = b(x + y) - c(x)$$

where b is the benefit to both players and depends on the sum of the two investments, and c is the cost to the player and depends on his own contribution. Clearly, if the players invest differently, the one who invests less does better.

Now the objective of the game is to get as large a payoff as possible, but of course, this is complicated because the payoff depends on both strategies. To get a feeling for things, we explore the game with the benefit and cost functions: $b(z) = z(4 - z)$ and $c(x) = x^2$.

1. Suppose that I know the contribution y of my partner. Find my optimal contribution x^* in terms of y.

2. Suppose that the two players are a female A and a male B. The structure of the game is that A has to go first; that is, A chooses her strategy before B chooses his, so that when B makes his choice, *he knows what A has chosen*. The interesting question is: who gets the larger payoff, A or B? Investigate this using the b and c functions given above.

This is a nice problem, and, in a small group format, with occasional guidance from me, the students all manage to solve and understand the problem. They find that A has the higher payoff. It is an advantage to go first.

Then comes the homework: Suppose that the benefit function is increasing with diminishing returns (*i.e.*, $db/dz > 0$ and $d^2b/dz^2 < 0$) and the cost function is increasing and accelerating (*i.e.*, $dc/dx > 0$ and $d^2c/dx^2 > 0$). Show analytically that the result we obtained in class, that A does better than B, holds in this general situation.

How do you think the students did? Should students with two years of honours calculus be able to solve this problem given a week working together in groups? I had thought so. But in fact, no one managed that feat. Most seemed to have no clue how to go about it — how to "play" with it.

Specialist High School Major

Some time ago, I read in a local newspaper that the Catholic District School Board of Eastern Ontario was offering the *Specialist High School Major* (SHSM) program. My curiosity piqued, I made further enquiries and discovered that this is a recent initiative of the Ontario Ministry of Education. One of the goals of the Liberal Government is to discourage students from dropping out of high school and narrowing their career prospects. However, such a policy can be effective, only if the range of options offered to students extends beyond the standard college and university preparation material. The SHSM program seems to fill the bill quite nicely.

Partnered with local businesses and other community entities, as well as nearby colleges and universities, schools can provide opportunities for students to obtain preparatory knowledge and on-the-job experience in a number of practical areas, get some certifications and explore the Ontario Youth Apprenticeship Program (OYAP) and School-College-Work initiatives. In the current academic year, SHSMs are being offered in these sectors: Agriculture, Arts and Culture, Business, Community Safety and Emergency Services, Construction, the Environment, Forestry, Health and Wellness, Horticulture and Landscaping, Hospitality and Tourism, Information and Communications Technology, Manufacturing, Mining and Transportation. Students can be certified in such things as Workplace Hazardous Materials Information Systems, standard first aid and CPR, and customer service.

The academic requirement is a bundle of 8-10 Grade 11 and Grade 12 credits that include 4 major credits specific to the sector, 2-4 credits from the Ontario curriculum in which some expectations