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Nicholas H. Wasserman *Editor*

Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers

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Chapter 20
Blue Skies Above the Horizon

Ami M. Mamolo and Peter D. Taylor

Introduction

> How can secondary teacher education approaches, ones that use abstract algebra as it might be applied in secondary teaching situations, inform the way we think about connecting abstract algebra to secondary mathematics, for secondary mathematics teachers?

This is the question we were asked to consider when framing our commentary to the chapters included in this section of the volume, *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers*. As we discussed this question, and how we might address it, we found ourselves posing further questions, and we frame our commentary around those. Specifically, we will focus on the following two broad questions:

1. *Connections*. In what ways does abstract algebra connect to secondary school mathematics and how can we understand these connections in terms of teachers' disciplinary knowledge?
2. *Approaches*. How can we then build on these connections to support the development of teachers' disciplinary knowledge?

A. M. Mamolo (✉)
Faculty of Education, University of Ontario Institute of Technology, Oshawa, ON, Canada
e-mail: ami.mamolo@uoit.ca

P. D. Taylor
Department of Mathematics and Statistics, Queen's University, Kingston, ON, Canada

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Connections

Each of the chapters presented in this section exemplifies ways in which abstract algebra may be connected and applicable to secondary school mathematics teaching, with a particular emphasis on connections amongst mathematical content and practices. The authors present different perspectives for why they promote their particular connections: (1) to make the work of teachers easier, (2) to enrich mathematical understanding for pupils, (3) to support teacher responses to contingency, and (4) to highlight relationships amongst mathematical ideas. Somewhat implicit in all of this are the authors' underlying intentions for secondary school mathematical experiences. We have our own views on this, and we articulate some of them here, to help contextualize our discussion of the ideas raised within each chapter. As teacher educators, the overarching goals we hold for secondary school mathematics inform our priorities when structuring experiences for prospective secondary teachers. How we support prospective teachers in developing practices that are in line with our goals relates to our understanding of teachers' disciplinary knowledge, and vice versa. Our understanding of teachers' disciplinary knowledge informs the support we can give prospective teachers in developing practices that are in line with our goals. Thus, our response to the question "in what ways does abstract algebra connect to secondary mathematics and how can we understand these connections in terms of teachers' disciplinary knowledge?" is broken down into the following subsections:

- Goals for secondary mathematics: Blue skies and feet on the ground
- Teachers' disciplinary knowledge: A view of mathematical horizons
- Analyzing connections

Goals for Secondary Mathematics: Blue Skies and Feet on the Ground

A detailed discussion of our goals for secondary mathematics would take us well outside the limits of this chapter, so instead, we offer highlights. First and foremost, our position is that secondary students should be engaging with meaningful, relevant, interesting, and challenging activities and problems. We acknowledge that this is a bit of a "blue sky" goal—many teachers only rarely are able to tackle problems of that kind. Administrators, parents, and students can be reticent toward approaches that diverge from established routines and practices, and too often those practices rely on bite-sized exercises that reinforce rules, processes, and calculations. Nevertheless, our blue sky goals are motivated by pragmatic considerations—we want students to appreciate mathematics, its structure, its relevance, its beauty, and how it connects to their scholarly, professional, and personal life trajectories. For this to work, their teachers must also have such an appreciation.

There are challenges. Disparities continue to exist amongst mathematical practices engaged in at school and mathematical practices needed by mathematicians, those that are applicable to various professions, and those that are valuable for informed citizenry. Cuoco, in his introduction (Chap. 18), speaks of a career-long personal agenda to find ways to close the (huge) gap between school mathematics and mathematics as it is practiced by mathematics professionals. Boaler (2016) and Taylor (2018) both declare that the examples we work with in class should be ones that are of interest to a mathematician—and we interpret this broadly to include various "types" of applied mathematicians. This view, and more general versions of it, goes back a long way, for example, to Whitehead (1929) and Dewey (1934), both of whom emphasized the quality of the student experience. This draws our focus much more to doing rather than knowing. Or if you like, knowledge is wonderful, provided you are doing something interesting with it, which is not always the case in school mathematics.

Let us state an operational version of this. Whenever we, as teachers of mathematics, meet with students at any level, we should bring, along with the knowledge we are offering, an activity or problem that is interesting and that has meaning in the students' lives. Papert (1972) makes this point:

> The important difference between the work of a child in an elementary mathematics class and that of a mathematician is not in the subject matter (old fashioned numbers versus groups or categories or whatever) but in the fact that the mathematician is creatively engaged in the pursuit of a personally meaningful project. In this respect a child's work in an art class is often close to that of a grown-up artist (p. 249).

The analogy between mathematics and art is compelling to us, and we note that several researchers have emphasized the creative and aesthetic character of mathematics (e.g., Barabe & Proulx, 2017; Boaler, 2016; Gadanidis, Borba, Hughes, & Lacerda, 2016; Raymond, 2018; Sinclair, 2006; Taylor, 2018). For us, this reaches back to the theme of Dewey's (1934) *Art and Experience*, that the aesthetic experience is jointly constructed between painter and viewer, performer and audience, that both are called to be artists in a shared experience. We imagine a parallel ideal, where both teacher and student are called to be mathematicians in a shared experience of doing interesting mathematics. To realize such an ideal, teachers need to have the courage and the knowledge to be more independent in how they interpret and enact curricula, so as to work with material that really interests and excites them and their pupils. They need an appreciation for the nature of mathematics, its structure, its relevance, its beauty. We see abstract algebra as a wonderful context in which to foster such an appreciation, and one that has important implications for teachers' disciplinary knowledge.

Teachers' Disciplinary Knowledge: A View of Mathematical Horizons

Mathematical knowledge required for teaching has been widely discussed, with attention focusing on knowledge in teaching, for teaching, and of teachers (e.g., Adler & Ball, 2009; Ball, Thames, & Phelps, 2008; Davis & Simmt, 2006; and many, many others). As part of their extensive work on teacher knowledge, Ball and colleagues introduced the construct of Knowledge at the Mathematical Horizon (KMH), which was described as knowledge which "engages those aspects of the mathematics that . . . illuminate and confer a comprehensible sense of the larger significance of what may be only partially revealed in the mathematics of the moment" (Ball & Bass, 2009, p. 5). While there is no consensus on how to define KMH, it has been characterized by key elements of a teachers' mathematical knowledge that extend beyond curricular content, such as, knowledge of mathematical structures, practices, and values (Ball & Bass, 2009). Evidence suggests that KMH is an integral part of teachers' disciplinary knowledge, and it has been positively linked to teachers' abilities to respond in the moment to classroom interactions (Fernandez & Figueiras, 2014; Jakobsen, Thames, Ribeiro, & Delaney, 2012; Zazkis & Mamolo, 2011), to plan and extend lessons (Wasserman & Stockton, 2013), and in anticipating and responding to student learning (Mamolo & Pali, 2014).

Our view of the horizon aligns with the perspective of Zazkis and Mamolo (2011), who suggest that KMH is intimately related to a teacher's focus of attention and his or her ability to flexibly shift attention, such that relevant properties, generalities, or connections, which embed particular mathematical content in a greater structure, are accessed in teaching situations. Specifically, KMH is conceptualized as a teacher's knowledge of the horizon of a mathematical object, and draws on Husserl's philosophical notions of inner and outer horizon (Follesdal, 2003). That is, when an individual attends to an object, he or she will focus on particular features of that object—e.g., if you think of a circle, your attention might be focused on its size and shape—while other features of that same object will lie in the periphery—e.g., the equation of that circle, or its position in space. These peripheral features, all of which are specific to the object of thought, lie within the object's inner horizon—they are "aspects of an object that are not at the focus of attention, but that are also intended" (Zazkis & Mamolo, 2011, p. 9). Further in the periphery are "features which are not in themselves aspects of the object, but which are connected to the world in which the object exists" (*ibid*, p. 9), and as such comprise the object's outer horizon. With respect to our circle, the outer horizon includes structures, such as trigonometric identities, conics, and geodesics, as well as ways of working mathematically with circles.

Metaphorically, knowledge of the horizon depends on our location in the terrain. A high vista can offer a broad view of the terrain—where we've been, where we can go, ways we can get there, and potential obstacles along the way. In contrast, anyone who has ever been lost in the woods knows the challenges of navigating your way without such a view. Zazkis and others have argued that

Table 20.1 Interpreting components of KMH

KMH components (Ball & Bass, 2009)	Our interpretation of components	Connections to inner and outer horizons
Mathematical environment surrounding current "location"	Knowledge of how the current subject matter relates to previously learned and future concepts, within and across specific grades	Is influenced by focus of attention (inner horizon) and understanding of the lay of the land (outer horizon)
Major disciplinary ideas and structures	Knowledge of the underlying structural components of mathematics, such as connections between seemingly disparate content	Structure embeds specific content within the greater mathematical world (outer horizon)
Key mathematical practices	Including conjecturing, generalizing, and proving	Ways of engaging in specific practices (inner horizon) within a greater mathematical world (outer horizon)
Core mathematical values and sensibilities	Including precision, axiomatic thinking, and questioning conventions	Ways of being within the greater mathematical world (outer horizon)

studying advanced mathematics can help teachers acquire a broad view by providing access to these "higher vistas." Our conceptualization of KMH illustrates how advanced mathematical knowledge can influence teachers' view of the terrain through a connection we make to the components of horizon knowledge introduced by Ball and Bass (2009). Specifically, Ball and Bass (2009) describe four major components of KMH: (1) a sense of the mathematical environment surrounding the current "location" in instruction; (2) major disciplinary ideas and structures; (3) key mathematical practices; and (4) core mathematical values and sensibilities. These components connect to the notions of inner and outer horizons, as described in Table 20.1. These connections speak most clearly to instances when the object of thought corresponds to mathematical content (e.g., circles, functions), however similar connections exist when the object of thought is more abstract (e.g., proving, recursive thinking, axiomatic thinking). In particular, we observe that knowledge of an object's outer horizon can influence what about that object is in view and what lies in the periphery (inner horizon). This, in turn, influences the view of the mathematical landscape, which embeds that object within a greater mathematical world (outer horizon), as well as provides a map for "instructional locations."

Applying these ideas to examples from the chapters allowed us to analyze some of the ways in which abstract algebra can be connected to, and is applicable for, secondary mathematics teaching. We highlight three examples here—depicted in Figs. 20.1, 20.2, and 20.3—and then discuss further connections in the following subsection.

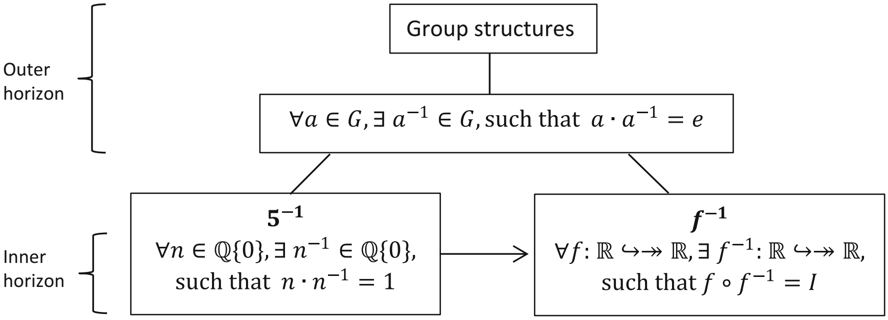

Fig. 20.1 Connecting examples of inverses via structures in the horizon

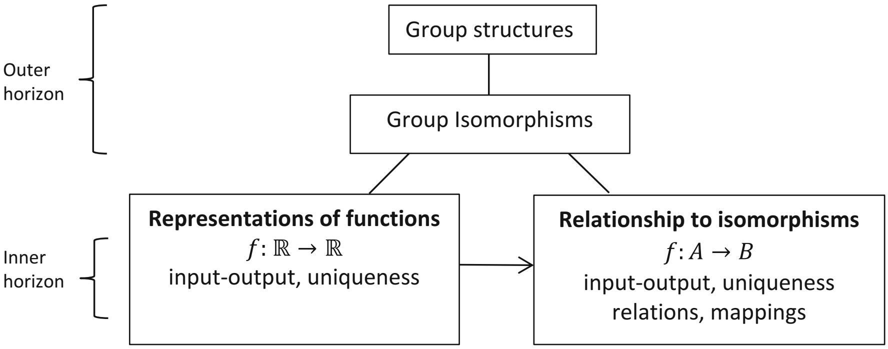

Fig. 20.2 Shifting focus from representations to relationships via broadened horizons

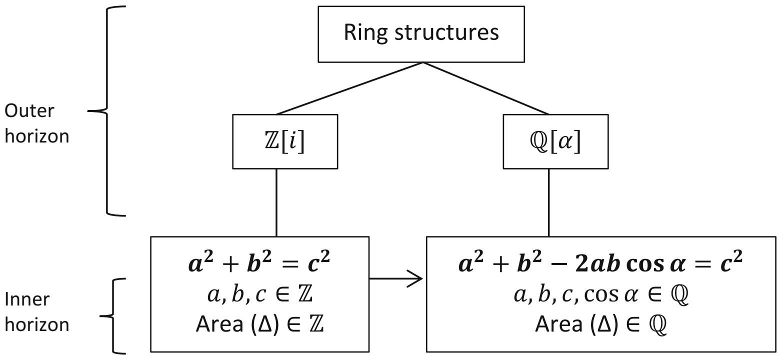

Fig. 20.3 Connecting Pythagorean triples with Heron triangles via structures in the horizon

Figure 20.1 offers an example of how KMH embeds specific examples in a broader context via knowledge of group theory. The example comes from Zazkis and Marmur (Chap. 17), who noted that knowledge of group theory is helpful in appreciating "reciprocal," as a specific instance of "inverse," as it draws attention to the structural similarities of the two. This is in contrast with their observation that many prospective secondary teachers view the concepts as distinct and context-specific. The analysis in Fig. 20.1 focuses on the number 5^{-1}, with some of its intended features lying in the peripheral inner horizon, including, for example, that 5 is a rational non-zero number, its reciprocal exists, and if you multiply 5 by its reciprocal, the product is 1. In the outer horizon are the group-theoretic structures which embed this specific instance in a more general and abstract setting, including the closure property of groups, and the existence of inverse and identity elements. These structures overlap with the outer horizon of f^{-1} and form a pathway for connecting the two examples. In the periphery or inner horizon of f^{-1}, we have depicted what we believe would be likely intended by secondary teachers, namely that the function is a bijection which acts on real numbers.

Figure 20.2 also considers functions and illustrates how KMH can occasion a shift in attention such that different properties of an object fall into view. The example comes from Wasserman and Galarza (Chap. 16), who characterized changes in secondary teachers' portrayal and emphases of functions after their engagement with the Function Module. Prior to the module, teachers emphasized multiple representations of functions (e.g., equations, tables, graphs) and restricted their examples to numerical ones. Figure 20.2 positions representations of functions at the focus of attention, and includes, in the inner horizon, the restriction of f to the set of real numbers and its "every input has a unique output" characterization. The Function Module introduced group isomorphisms and seemed to broaden teachers' awareness of how to think about functions within a greater context. That is, they drew on more abstract examples of functions, they positioned functions as a special kind of more general relations, and they stopped attending to representations of functions, and started attending to relationships between functions and isomorphisms. We highlight this shift in attention, but are careful not to place a value-judgment on it. It is not clear from the chapter whether a shift in attention from function representations to relationships with isomorphisms is a useful shift, even if it is an applicable shift. The usefulness would depend on how the material was enacted with students, and we return to this idea later on.

Our third example, depicted in Fig. 20.3, stems from Cuoco's rich and interesting exploration of Pythagorean triples and Heron triangles. In his Chap. 18, Cuoco develops a method for solving "one of the oldest meta-problems [which] involves the search for Pythagorean triples" (p. 385), and then modifies it to produce Heron triangles. The method abstracts from the particulars of the meta-problem to address it via ring theoretic structures, which are themselves at different levels of abstraction. Cuoco introduces ring structures from $\mathbb{Z}[i]$ to address the search for Pythagorean triples. Contextualizing the ring $\mathbb{Z}[i]$ in a more general and abstract setting allows connections to be made to other contexts. This allows Cuoco to construct the ring $\mathbb{Q}[\alpha]$, whose structures are analogously applied in the search for

Heron triangles. Figure 20.3 illustrates how KMH can forge links between different areas (and grades) of curriculum, via knowledge of ring theory.

Analyzing Connections: Wherefore, Abstract Algebra?

In the preceding subsection we analyzed examples from the chapters through a lens of KMH in order to illustrate how abstract algebra knowledge can inform a teacher's understanding of secondary school mathematics. It is worth questioning whether other areas of advanced mathematics would be similarly applicable, and we suggest that in many cases, more mathematics would not lead to better mathematics. Framing the discussion in terms of KMH helped us articulate for ourselves what in particular about abstract algebra makes it stand out in a crowded chest of mathematical treasures. Zazkis and Marmur suggest that it is precisely the abstract nature of algebraic structures which is so applicable: “While abstraction is regarded as a source of difficulty, it nonetheless possesses the potential to make the algebraic construct relevant and applicable for a large variety of concrete examples and mathematical topics” (p. 365). More than that though, it is structure itself that really stands out for us. It is the explicit attention to structure (e.g. of groups, fields, rings) that is of central importance in the study of abstract algebra—we compare and contrast structures, exemplify and extend them, investigate implications, push boundaries, and play with relationships, all with a sort of directness and cohesion that are not so clearly visible in other areas, such as real analysis.

In the language used to develop KMH, one could say that in abstract algebra, major disciplinary structures become the focus of attention, and key mathematical practices, values, and sensibilities are developed through engaging with these structures. If clear connections to secondary school mathematics can be made (either by or for the individual), then enriched understandings of the mathematical horizon can emerge. Knowledge of abstract algebra can:

- Enrich an object's outer horizon by linking that object to major disciplinary structures and ways of working with them
- Broaden an object's inner horizon by shedding light on previously unknown or unacknowledged properties of the object
- Occasion a shift in attention such that different or more general properties of the object come into view

The applicability of horizon knowledge in general and abstract algebra knowledge in particular, depends on how an individual connects this knowledge to teaching situations. We highlight some connections amongst abstract algebra content and secondary school content, as well as connections amongst abstract algebra understanding and decision-making in teaching situations.

Connections Amongst Abstract Algebra Content and Secondary School Content

A quick review of select curricula shows some direct connections between abstract algebra content and secondary content—vectors, their properties, and performing operations on them are included in, for example, the US Common Core Standards (2010), The National Curriculum in England (2014), the Australian Curriculum (2012), and the Ontario Curriculum (2007), which is used in much of eastern Canada (curricula for central, western, and northern Canada do not address vectors). In looking for other content-specific connections, and given the prominence of functions in various secondary mathematics curricula, it was not surprising to find that three of the four chapters in this section identified ways that abstract algebra could be applicable to working with functions. Wasserman and Galarza identified binary operations and group isomorphisms as examples of functions, and noted that working with binary operations and group isomorphisms could foster "a broader understanding of function," which could in turn "help exemplify nuances within and boundaries around the idea of functions as more than merely symbolic rules" (p. 341). Zazkis and Marmur looked at inverse functions and seemingly invertible functions and noted that "an internalized recognition of the existence of an algebraic structure of mathematical objects (including objects outside the algebraic domain) may aid teachers in their thought process" (p. 373) when clarifying student confusion or designing tasks. They also note that a group theoretic understanding of secondary content "brings the discussion to a higher level of abstraction, where different ideas exemplify the same structure" (p. 371). Structural similarities between rings were exploited by Cuoco when considering fitting functions to tables; his discussion of Newton's Difference Formula and Lagrange Interpolation highlights how the reformulation of problems with abstract algebra structures can shed new light on familiar content. Group, ring, and field structures were also applied to solving equations, number properties and operations, and "meta-problems" involving the measures of triangles.

Connections Amongst Abstract Algebra Understanding and Teaching Decisions

In considering different ways in which the understanding of abstract algebra may connect to teaching situations, we restrict our attention to four aspects related to the mathematical work of teaching:

- *Planning* includes such things as unit and lesson preparation, structuring of courses, and establishing assessment approaches
- *Task design* includes the creation, development, or acquisition, of specific tasks for learning, consolidation, assessment, practice, and so on

- *Norm enactment* includes how the teacher embodies and fosters social and socio-mathematical norms in the classroom, such as expectations for justifying statements, posing questions, or precision in defining terms
- *In-the-moment responses* include teachers' reactions to unanticipated mathematical ideas or utterances offered by students

The applicability of abstract algebra to each of these aspects of teaching was well exemplified by the chapters in this section. In planning activities, there were considerations of which definitions to use, what emphases to place, and which properties to address (Chap. 16); in designing tasks, careful choices in numerical examples were made so as not to "cloud the underlying method" (Chap. 18, p. 384); norms related to problem solving became more salient (Chap. 19); and in-the-moment responses were shaped and influenced by a sense of underlying structure (Chap. 17). Further, these four aspects of the mathematical work of teaching are all clearly interconnected, and each one can influence all of the others. For instance, the norms recognized and valued by a teacher can influence the planning and design of course materials in terms of what and how content is addressed, and affect how moments of contingency are addressed. Murray and Baldinger speak to such interconnectedness with their discussion of how an abstract algebra workshop helped teachers value the importance of precise language in defining concepts, justification and argumentation. The participants in this study were given "the mental space" to consider how valuing precision in their teaching "could impact students' understanding of solving equations" (p. 424). Engaging in workshop activities exposed teachers to the "possible strategies they might employ to strengthen student understanding of solving equations" (p. 425) in their lesson planning, task design, and in response to student questions. In-the-moment responses can serve as catalysts for rethinking planning and design, while also promoting and fostering normative standards of the classroom. Zazkis and Marmur provide an example of how an unexpected student question triggered a reconsideration of the original task and a redesign of the lesson for future teachers. They state: "In both the original and adapted version of the task, basic properties of group theory served as a guide for the instructor's responses and as a basis for further mathematical inquiry" (p. 371).

Approaches

Following the claim of Ball and Bass (2009, p. 11) that "we do not know how horizon knowledge can be helpfully acquired and developed," we sought ideas from the chapters to inform our thinking about this issue. We have thus far presented our view that abstract algebra has the potential to be a useful (and we might add beautiful, fun, enticing) context through which horizon knowledge can be enriched. However, we have not yet addressed the question of "how," and for us this boils down to a look at teaching approaches. For this, we consider the different roles that faculties of education and departments of mathematics each play in preparing secondary teachers. This helps paint a picture of where teachers might reasonably acquire and develop their horizon knowledge. We then look at specific approaches exemplified in the chapters, and highlight similarities and differences in approaches, as they might apply to the different demographics of prospective and practicing teachers. We conclude this section and chapter by applying what we have learned and offer a set of examples that build off of ideas from the chapters of this section. Thus, our exploration of approaches draws on our discussion of ways that abstract algebra connects to secondary mathematics. Specifically, we address the question of "*how can we build on these connections to support the development of teachers' disciplinary knowledge?*" via the following subsections:

- A vision of teacher preparation: Where from, abstract algebra?
- Structuring educational approaches: Broadening horizons
- Extending connections: Blue skies above the horizon

A Vision of Teacher Preparation: Where from, Abstract Algebra?

Murray and Baldinger cite recommendations from the CBMS (2012) report that advocate for the inclusion of courses in advanced mathematics, such as abstract algebra, in the preparation of future secondary mathematics teachers. They note further that "among four-year institutions with secondary pre-service teaching certification programs, 89% of all mathematics departments require their students to take abstract algebra" (p. 403). The implication seems to be that there is recognition that abstract algebra "can sure help" teachers, as Zazkis and Marmur quipped. However, many teacher education programs do not include content-specific requirements for admission, nor does the pool of secondary teacher candidates include students only (or perhaps even mostly) from departments of mathematics. In Canada, for example, a secondary mathematics teacher candidate can gain entry to a bachelor of education program with only three 6.0 credit mathematics courses from their undergraduate studies, and these courses tend to be Calculus I and II, and Statistics. It is important to note that the content and teaching approaches that are common, and even possible, in (typically) large-scale calculus and statistic courses, are significantly different from what is common or possible when studying abstract algebra. Further, there seems to be little impetus for prospective teachers to study abstract algebra in university: calculus and statistics have more apparent connections to school curricula, they have "friendlier" reputations amongst students, and there are often more supports for student learning in these courses. Thus, we agree with Zazkis and Marmur that "it is reasonable to assume that future teachers studying at university are not likely to focus on group theory as an important topic in support of their future career" (p. 365).

We mention this because one of the first reactions we had to the examples discussed in these chapters was that, for better or worse, very few of the problems, activities, or approaches illustrated would be appropriate in a bachelor of education program. The typical focus on "methods" in faculties of education almost precludes the introduction of new (advanced, non-curricular) content, even in content-focused applications, such as analyzing student error patterns. In our experiences with colleagues from various faculties of education, it has been fairly standard to find attitudes ranging from indifference to open hostility toward advanced mathematical knowledge. As such, we suggest that making space for abstract algebra in a bachelor of education program would be a tough sell. Indeed, the authors of this section's chapters might agree with us: the modules discussed in Wasserman and Galarza were developed for a masters-level abstract algebra course, Cuoco's examples stemmed from years of teaching experiences and a personal affinity for mathematics encountered during graduate studies, Murray and Baldinger developed content-specific workshops for practicing teachers, and Zazkis and Marmur exemplified how a teacher educator's knowledge of abstract algebra can help direct prospective teachers' attention to important mathematical structures within school curriculum.

It seems to us that the departments of mathematics might be a more hopeful place where teachers could be exposed to abstract algebra concepts and their connections to secondary school teaching. We recall a conversation with a colleague who taught a masters-level mathematics course called *Abstract Algebra for Teachers*. When asked how "abstract algebra for teachers" was different from "abstract algebra," the response was "Actually, I hadn't thought about that." So, we thought about it. And in reflecting on our own experiences with the subject matter, and with the ideas raised in the chapters, we suggest that first of all, there needs to be a difference, and secondly, there ought not to be much difference at all!

The need for a difference comes from a difference in course objectives and a difference in learners' focus of attention. First, we look at course objectives: For the most part, courses in abstract algebra offered in undergraduate mathematics programs have mathematical content as a primary objective. Thus, they tend to treat topics in a comprehensive manner and move with reasonable speed through the material. Their choice of topics is also significantly influenced by areas of current research interest; for example, the beginning course that all mathematics majors take, typically offered in the second or third year of undergraduate studies, focuses on rings and fields, leaving group theory to later more specialized courses that many majors do not take. On the other hand, a course aimed at teachers could have the development of mathematical thinking and structured play as primary objectives, and comprehensive coverage would not be so important. Indeed, one could imagine a course with a collection of wonderful activities around groups and rings that would *not* start students on the road to a PhD in mathematics, but *would* give them a sense of the power of working with structure, of using aesthetic principles as a guide to the way forward (Sinclair, 2006), and most importantly, how guided play with concrete objects can lead to an understanding of abstract structures.

Having said all of that, the more we interact with undergraduate students in an honours mathematics program, both while they are students and after they graduate, the more we have come to believe that the course that we have just described for teachers would be right for the capacities and needs of these students as well. This is certainly the case for those students who wind up in the general business/industry environment, including the specialized STEM areas, but we feel that it would also be the case for those few who *do* go on to a PhD in mathematics, as these students will presumably already have the capacity to extract theoretical results from mathematically rich particular examples. The key here is to use "low-floor, high-ceiling" activities (Gadanidis et al., 2016; Boaler, 2016) that give all students welcome access and invite more ambitious students to probe more deeply.

To be clear, the courses we are suggesting here are about mathematics, not pedagogy. It is their pedagogical style, focusing on mathematical thinking and investigation, that will set them apart from most of what is currently offered in programs for math majors. We feel that all students would welcome that type of course. Having said *that*, experience with our colleagues in mathematics departments has led us to feel that they may resist such a change in pedagogy. Although wonderful exceptions exist, a comprehensive linear development remains the normative choice for both textbooks and classrooms.

Regarding learners' focus of attention, we have in mind two possible ways teachers may study abstract algebra in support of their career—as prospective teachers in an undergraduate mathematics course, or as practicing teachers in a graduate mathematics education course. In the former case, the primary learning objective would likely align with the primary course objective of the undergraduate program and focus on mathematical content. Thus, their attention would be on their own personal scholarly growth. In the latter case, the primary learning objective of the teacher-student is not solely on personal scholarly growth, but also includes the scholarly growth of their students. In other words, the two sets of learners are differently motivated in their choice of courses. These differences have implications for educational approaches in abstract algebra, as we discuss in the next section.

Structuring Educational Approaches: Broadening Horizons

In our work with prospective teachers, we have seen differences in how they tackle a problem. Some will stare at it with little sense of how to begin, of how to even think of what needs to be done. Others seem to be able to see that "this is a special case of that," or "this belongs over there," or "if I am to show this, I'm going to have to show that first." These students have a grasp of the structure of the problem that allows them to start moving. Recalling metaphors for horizon, it's analogous to finding yourself in the middle of a large unfamiliar city with instructions that you are to be somewhere at a certain time. Having an aerialmap makes a huge difference, particularly if you have experience in reading the map, for example, in how the different modes of potential transport are color-coded, and how you can transfer from one to the other. We suggest that the study of abstract algebra can help develop such a map for the mathematical landscapes of secondary school and further, can foster the ways of working with that map to understand how to get around. That is, the study of abstract algebra can develop the capacity to identify and work with the structure of a problem, and this is a valuable component of a teacher's KMH.

When examining the chapters in this section for ideas of how to structure educational approaches for learning abstract algebra, we found both explicit and implicit suggestions. Cuoco's chapter, although it did not specifically address educational approaches, nevertheless exemplified important considerations. Through the author's own rich view of the horizon, we note an importance in understanding definitions and properties, as well as in connecting different methods to solve the same problem. Cuoco's understanding of mathematical structure is exemplified in his comparisons of algebraic and geometric methods, as well as in connecting methods at different levels of abstraction. For each idea addressed, it is clear that he has a sense of "where this sits in the bigger picture" (p. 393). We found many good problems in this chapter that could be very useful, depending on how they are framed. More on this idea in a bit.

Of the educational approaches explicitly addressed in the chapters, we note the importance of Mason's (2002) "experience of disturbance," as described by Zazkis and Marmur. A disturbance can compel an individual to rethink previous knowledge, to seek out new ideas, and it can elicit a shift in attention, and can enhance awareness—in short, it can broaden horizons. We also appreciated the emphasis placed on question-posing by Murray and Baldinger. They developed questions to explicitly draw teachers' attention toward the similarities and differences of secondary mathematics and abstract algebra. Explicit connections amongst secondary mathematics and abstract algebra were also advocated for by Wasserman and Galarza. We suggest that explicit attention to these connections is valuable for both undergraduate mathematics students, as well as practicing teachers, but for different reasons. The former group would benefit from the opportunity to build on what they have already learned, and it could go a long way for making the material more accessible, without necessarily hampering the agenda of professors hoping to educate the next generation of abstract algebra researchers. The benefits for practicing teachers were discussed by Wasserman and Galarza, who drew on the instructional model developed by Wasserman, Fukawa-Connelly, Villanueva, Mejia-Ramos, and Weber (2017) and Wasserman, Weber, and McGuffey (2017) in the design of their modules. The instructional model "is composed of two parts: building up from and stepping down to practice" (p. 338). The authors elaborate that, "[i]n between building up from and stepping down to practice, the advanced mathematics topics are taught by the instructor in ways true to its advanced nature with formal rigorous treatment" (p. 338). Such a model intends to help teachers appreciate the relevance and role of advanced mathematical knowledge in their teaching practice.

The need to consider how we frame abstract algebra problems, particularly for practicing teachers, relates to the teacher-students' learning objectives mentioned above: their interest is shared between personal growth and the potential to foster their students' growth. Thus, we suggest that what will compel teachers to learn abstract algebra will be different from what will compel undergraduate students, and as such, problems need to be framed differently. Perhaps most significantly, there will be a difference in what might constitute a disturbance of experience – for practicing teachers, that disturbance will likely need to come from classroom practice.

Extending Connections: Blue Skies Above the Horizon

In concluding this commentary chapter, we repeat our main premise: that abstract algebra can provide a wonderful context in which to develop and nurture a genuine appreciation of mathematics. Our engagement with material from the chapters in this section elicited a broadening of our horizons and inspired some blue-sky thinking about valuable ways to engage with mathematics that connect to and extend some of the examples presented. We use this opportunity to offer instructional ideas, which are motivated by our desire to foster interesting, meaningful, and valuable mathematical engagement for secondary teachers.

Order of Operations

Zazkis and Marmur broach this topic. It is a huge idea in mathematics, and indeed the whole notion of linearity, $f(a + b) = f(a) + f(b)$, is about interchanging order of operations, as is the fundamental theorem of calculus. In the task Zazkis and Marmur present, they discuss expressions, all alternating multiplication and division, such as

$$a \div b \times c \div d$$

which were given to prospective teachers. The teachers were told that a student did the calculation by computing the two divisions first, followed by the multiplication, and were asked whether this was correct. We felt that this task could make mathematics seem more confusing and harder than it really is, though our concerns might have been assuaged if we had been provided with more details about exactly how the task was scaffolded and what discussions might have preceded it. Nevertheless, the task offers an interesting context in which to foster important mathematical sensibilities, such as recognizing, challenging, and exploring within the constraints of mathematical conventions. The expression can invite one to first ask what the conventions are and second to explore how many different answers might be obtained with all possible conventions. We connect this to the kind of mathematical thinking that Mason (2001) refers to as searching for freedom within constraint. This could lead to rich mathematical discussions, as well as motivate why mathematicians generally avoid the use of the symbol ÷, preferring that students get into the habit of writing expressions such as

$$\frac{a}{b} \cdot \frac{c}{d}$$

where the conventions are clear.

Seemingly Invertible Functions

Another task presented by Zazkis and Marmur defined a pair of functions f and g from $\mathbb{R}$ to $\mathbb{R}$ to be "seemingly invertible" if

$$f\,(g(x)) = x$$

for every x in $\mathbb{R}$. The question given to the teachers was whether this implied that g is invertible. This task certainly has mathematical meaning and could potentially connect with a number of different kinds of functions and transformations. Similar to their other example, we found ourselves wondering more about the details of how the authors engaged teachers in the investigation. For example, in our first reading of the paper, we were unsure how we might manage to get the teacher candidates that we have experience with, to come up with the exponential-logarithm example, as it seems to be unintuitive at first. More generally, our feeling is that teacher candidates, along with most graduates of first-year calculus courses, are uncomfortable "playing" with functions of a real variable.

Here we give an example of how we might envision the scaffolding of the task. We would first make sure that the students knew what an invertible function was, and being wary of their facility with formal notation, we would ask them to present the definition with a simple diagram and a finite domain. We would expect a diagram such as Fig. 20.4.

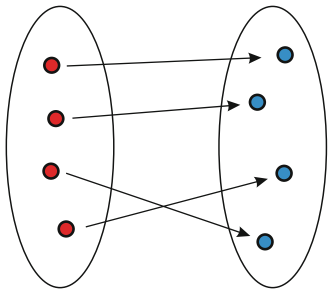

Fig. 20.4 A simple example of an invertible function

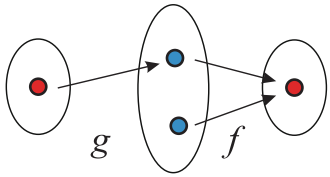

Fig. 20.5 The simplest example in which $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})) = \boldsymbol{x}$ but $\boldsymbol{g}$ is not invertible

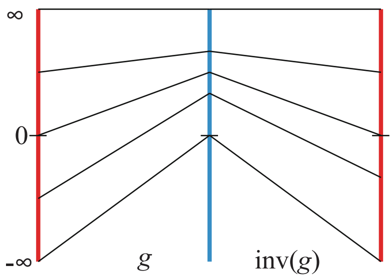

Fig. 20.6 A generalization of Fig. 20.5 to real number line domains

Then, we would ask them to explore the question of the invertibility of g using the arrow representation with the smallest possible domain sets that give an example in which $f(g(x)) = x$ with g not invertible. We would expect them to come up with the counterexample in Fig. 20.5.

Of course our final objective is to explore the example on the real number line and an interesting task at this point would be to ask the students to adapt the setup of Fig. 20.5 to the case in which both domains would be $\mathbb{R}$. We feel that this could lead to an interesting small group activity. It seems clear that the two blue points in the middle set would need to be "half" the real number line each, while the red point would be the entire line. We expect that this would lead to the diagram of Fig. 20.6.

This diagram might well lead the students to the exponential and logarithmic forms, possibly via the square and square root functions, as discussed in the chapter. Of course, we still have the question of what f should do with the negative part of the middle domain and it might easily be seen from this setup that it doesn't really matter.

Math and Music

Wasserman and Galarza present a task that exploits the observation that the keys on a piano work in a *mod* 12 manner, and they introduce the "distance from *C*" operation as a context in which to discuss isomorphic groups. This example seemed to us to be a bit limited, and perhaps contrived, in that it missed some important and natural learning opportunities.

What is significant about the remarkable number of interactions between mathematics and the real world is that they have the capacity to give us new and powerful insights into the structure of the phenomenon being modeled. This is certainly the case for the structure of the musical scale and there are a number of activities we might give students that then could provide insights into this structure. In Fig. 20.7,

a

0	1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80	81	82	83
84	85	86	87	88	89	90	91	92	93	94	95
96	97	98	99	100	101	102	103	104	105	106	107

The integers arranged in mod(**12**) columns.

b

C	C#	D	E♭	E	F	F#	G	G#	A	B♭	B
16.35	17.32	18.35	19.45	20.60	21.83	23.12	24.50	25.96	27.50	29.14	30.87
32.70	34.65	36.71	38.89	41.20	43.65	46.25	49.00	51.91	55.00	58.27	61.74
65.41	69.30	73.42	77.78	82.41	87.31	92.50	98.00	103.8	110.0	116.5	123.5
130.8	138.6	146.8	155.6	164.8	174.6	185.0	196.0	207.7	220.0	233.1	246.9
261.6	277.2	293.7	311.1	329.6	349.2	370.0	392.0	415.3	440.0	466.2	493.9
523.3	554.4	587.3	622.3	659.3	698.5	740.0	784.0	830.6	880.0	932.3	987.8
1047	1109	1175	1245	1319	1397	1480	1568	1661	1760	1865	1976
2093	2217	2349	2489	2637	2794	2960	3136	3322	3520	3729	3951
4186	4435	4699	4978	5274	5588	5920	6272	6645	7040	7459	7902

The frequencies of the piano keys in cycles/second (using what is called the "even-tempered scale"). The standard piano range is A0 = 27.50Hz to C8=4186Hz. "Middle C" is usually taken to be at 261.6 Hz.

Fig. 20.7 (**a**) The integers arranged in *mod*(12) columns. (**b**) The frequencies of the piano keys in cycles/second (using what is called the "even-tempered scale"). The standard piano range is A0 = 27.50 Hz to C8 = 4186 Hz. "Middle C" is usually taken to be at 261.6 Hz

we give an example that works with the multiplicative structure of the piano keys. The two tables are not only the same size, but they also have a parallel arithmetic structure. First of all, there is a clear one-to-one correspondence based on position, that is 0 corresponds to 16.35 and 67 corresponds to 784.0. Let's calls this mapping the function F. Thus $F(67) = 784.0$ Some questions we could ask include:

1. What happens algebraically as you go along the rows of the frequency table?
2. What happens algebraically as you go down the columns of the frequency table?
3. Find a formula for $F(n)$ in terms of n.

In fact there is much treasure to be harvested from the questions posed in this example, particularly if there's a keyboard available. The point is that if we are going to bring music into the school classroom, we should bring it fully in, play with it, and seek to understand how it works. In many ways, that is what mathematicians do best[1].

[1] For those who might be interested in a more detailed discussion, the following link presents such an activity: http://www.mast.queensu.ca/~math9-12/musical%20magic%20of%2012.html.

Table of Differences

In his polynomial interpolation example, Cuoco starts with a sequence that might have emerged from a particular exploration, and shows how the successive-difference method can tell the student whether the sequence can be generated by a polynomial, and in this case, the number of steps until the differences are constant will tell us the degree of the polynomial. Many students have seen this constant difference argument, often in working with quadratic or cubic polynomials, but given these constant differences, some work remains in finding the coefficients of the polynomial. By carefully tracking these calculations, Cuoco produces an elegant formulation of the polynomial in terms of the combinatorial coefficients. He suggests that these coefficients often reveal “combinatorial treasures” hidden in the original example. We agree. In fact, we have used the well-known problem about the number of regions, R, in a circle, formed from all chords between n points on the circle (in which chords intersect at distinct points), with secondary school students. The investigation leads to $R = \binom{n}{0} + \binom{n}{2} + \binom{n}{4}$, which can be obtained nicely by Cuoco’s difference method, and indeed provides combinatorial treasures for discussion as well as opportunities to discuss inductive proof approaches.

References

Adler, J., & Ball, D. (Eds.). (2009). Knowing and using mathematics in teaching. *For the Learning of Mathematics, 29*(3), 1–56.

Australian Curriculum. (2012). Retrieved from: https://www.australiancurriculum.edu.au/senior-secondary-curriculum/mathematics/

Ball, D. L., & Bass, H. (2009). With an eye on the mathematical horizon: Knowing mathematics for teaching to learners’ mathematical futures. Paper presented at the 43rd Jahrestagung der Gelleschaft fur Didaktic der Mathematik. Oldenburg, .Germany. Retrieved May 15, 2011, from: www.mathematik.uni-dortmund.de/ieem/BzMU/BzMU2009/BzMU2009-Inhalt-fuer-Homepage.htm

Ball, D., Thames, H. M., & Phelps, G. (2008). Content knowledge for teaching. *Journal of Teacher Education, 59*(5), 389–407.

Barabe, G., & Proulx, J. (2017). Revolutionner l’enseignment des mathematiques: Le projet visionnaire de Seymour Papert. *For the Learning of Mathematics, 37*(2), 25–29.

Boaler, J. (2016). *Mathematical mindsets: Unleashing students’ potential through creative math, inspiring messages and innovative teaching*. San Francisco, CA: Jossey-Bass.

Common Core State Standards in Mathematics (CCSS-M). (2010). Retrieved from: http://www.corestandards.org/the-standards/mathematics

Conference Board of Mathematical Sciences (CBMS). (2012). *The mathematical education of teachers II*. Washington, DC: American Mathematical Society and Mathematical Association of America.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*(3), 293–319.

Department for Education. (2014). *The national curriculum in England: Key stages 3 and 4 framework document*. Retrieved from: https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/381754/SECONDARY_national_curriculum.pdf
Dewey, J. (1934). *Art as experience*. New York, NY: Putnam.
Fernandez, S., & Figueiras, L. (2014). Horizon content knowledge: Shaping MKT for a continuous mathematical education. *REDIMAT, 3*(1), 7–29.
Follesdal, D. (2003). Husserl, Edmund (1859-1938). In E. Craig (Ed.), *Routledge encyclopaedia of philosophy* (2nd ed.). London: Routledge. Retrieved July 19, 2010, from: www.rep.routledge.com/article/DD029
Gadanidis, G., Borba, M., Hughes, J., & Lacerda, H. (2016). Designing aesthetic experiences for young mathematicians: A model for mathematics education reform. *International Journal for Research in Mathematics Education, 6*(2), 225–244.
Jakobsen, A., Thames, M. H., Ribeiro, C. M., & Delaney, S. (2012). Delineating issues related to horizon content knowledge for mathematics teaching. Paper presented at the Eighth Congress of European Research in Mathematics Education (CERME-8). Retrieved from: cerme8.metu.edu.tr/wgpapers/WG17/WG17_Jakobsen_Thames_Ribeiro.pdf
Mamolo, A., & Pali, R. (2014). Factors influencing prospective teachers' recommendations to students: Horizons, hexagons, and heed. *Mathematical Thinking and Learning, 16*(1), 32–50.
Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.
Mason, J. (2001). Mathematics as a constructive enterprise. In J. Winter (Ed.), *Proceedings of the British Society for Research into Learning Mathematics (Vol. 21, Num. 3)* (pp. 52–57). Southampton: British Society for Research into Learning Mathematics.
Ministry of Education. (2007). *The Ontario curriculum grades 11 and 12: Mathematics*. Retrieved from: http://www.edu.gov.on.ca/eng/curriculum/secondary/math1112currb.pdf
Papert, S. (1972). Teaching children to be mathematicians versus teaching about mathematics. *International Journal of Mathematical Education in Science and Technology, 3*(3), 249–262.
Raymond, K. (2018). M is not just for STEM: How myths about the purposes of mathematics education have narrowed mathematics curricula in the United States. *Education Sciences, 8*(2), 47.
Sinclair, N. (2006). *Mathematics and beauty*. New York, NY: Teachers College Press.
Taylor, P. (2018). Teach the mathematics of mathematicians. *Education Sciences, 8*(2), 56.
Wasserman, N., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS, 27*(6), 559–578.
Wasserman, N., Weber, K., & McGuffey, W. (2017). Leveraging real analysis to foster pedagogical practices. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th annual conference on Research in Undergraduate Mathematics Education (RUME)* (pp. 1–15). San Diego, CA: RUME.
Wasserman, N. H., & Stockton, J. C. (2013). Horizon content knowledge in the work of teaching: A focus on planning. *For the Learning of Mathematics, 33*(3), 20–22.
Whitehead, A. N. (1929). *Aims of education*. New York, NY: The Free Press.
Zazkis, R., & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics, 31*(2), 8–13.