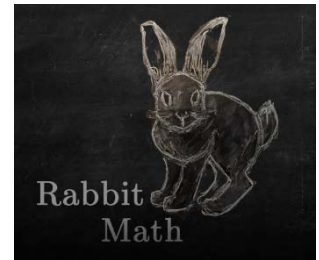


# RabbitMath



## A summary of our problems and activities.

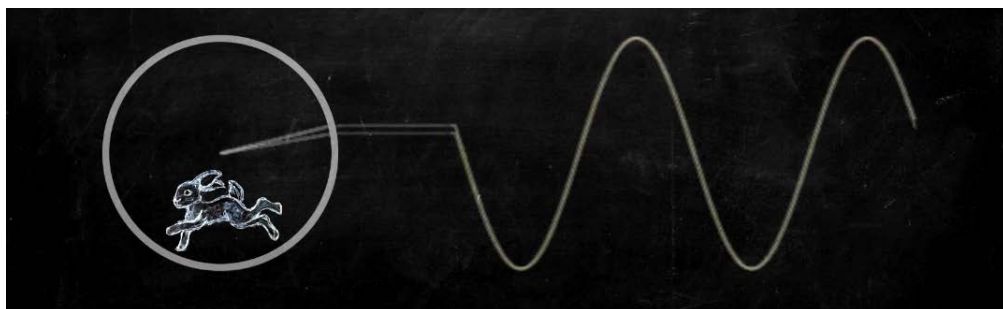
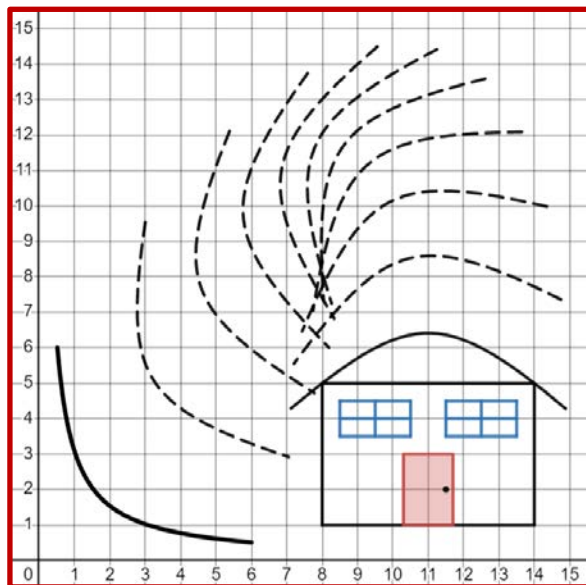
<https://www.rabbitmath.ca/>

These are designed for high-school students and teachers but they have all been chosen because students and teachers at the university level find them engaging. Most of them will work, at different levels of sophistication, at any of the four grade levels, and we have even tried some of them out in grades 7 and 8. The focus is not so much on mathematical knowledge as on mathematical thinking. In fact we have come to view our objective as teachers to be primarily on preparing our student to work with complex systems, as that's what we find that our math students need most of all.

Indeed, in our view, a student who had grappled seriously with all these problems would be ready for university math, almost no matter what "content" had or had not been formally covered.

We are not sure how to organize these problems as most of them can work at more than one grade level; the distinction that matters is often more about the level of sophistication than that of content. In that sense our view of the structure of the ideal math curriculum is a lot more like that of the English curriculum where there is a collection of literary works of all kinds that could be studied at different levels of sophistication. So we want you to simply browse and enjoy the landscape.

Our objective is to continue to add to this collection and to support the activities with teacher manuals, homework problems and videos.



## 8-cycle

Take any number  $n$  (a positive integer), square its digits, and add them up. What you get is a new number which we will call  $T(n)$ .

$T$  is a transformation that changes one number into another. Our objective here is to understand what happens when we apply that transformation again and again,

For example, suppose we start with 2021. Then applying  $T$  again and again:

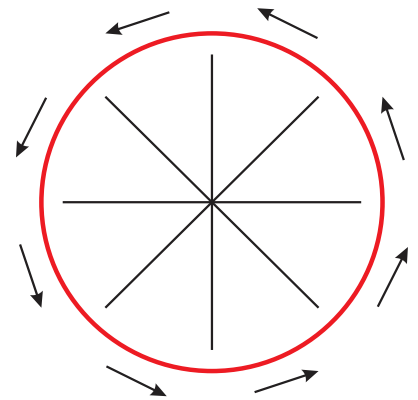
$$2021 \rightarrow 9 \rightarrow 81 \rightarrow 65 \rightarrow 61 \rightarrow 37 \rightarrow 58 \rightarrow 89 \rightarrow 145 \dots$$

and we keep right on going. What eventually happens? What happens for other starting numbers?

This is a fascinating problem. Right away the kids can start generating data. And they wonder how there could be a “universal” pattern. What makes this problem a real winner is that:

- A striking pattern is ultimately observed.
- There are many ways to make an argument that it always holds.
- The steps in the argument require careful persistent thinking.
- One of these steps requires some organized teamwork.

[8-cycle Teaching Manual](#)



## 100 Boxes

Here's the game. There are 100 players on the team and they are numbered from 1 to 100. The team has to come up with a strategy to win at the following challenge.

One by one, the players will be admitted into a room, player 1 first, followed by player 2, etc. In the room there is a long table with a row of 100 wooden boxes numbered in order from 1 to 100. Each box contains a card with a number on it such that each number between 1 and 100 appears on exactly one card. These numbered cards have been distributed randomly among the 100 boxes.



Once a player steps inside the room, he will have no further contact with any player who has not yet been in the room. While the player is in the room he is allowed to open and look inside 50 boxes—any 50 that he wants. Following that he must close the boxes leaving them in exactly the same state as he found them. After he has opened his 50 boxes he leaves the room. This goes on until all 100 players have passed, one by one, through the room.

The team will win the game if *every* player opens the box that contains his or her own number. They will lose if at least one player fails to find his or her number.

Find a strategy that maximizes the winning probability and find this probability.

Remarkably enough, there is a strategy which will give the team a win with probability greater than 30%. Even more remarkable, the strategy is elegant and simple. So simple it is easily understood.

I find that unbelievable.

[100 Boxes Teaching Manual](#)

One great thing about this problem is that we get a chance to delve in to the structure of permutations.

A bit of thought suggests that this is a rather hopeless challenge. A random strategy (players acting independently) will win with probability  $(1/2)^{100}$  a rather small number.

## Ant Cone

Here we are going to play with a circular cone of height 16 and base radius 4. First we get each student (or group) to construct a full-size model of the cone (it's a fun task) and then we introduce an ant that is going to crawl up and down on the cone.

First we set the cone down on a 4×4 base and ask how high it is at different points:

$$(2,0), (0,3) \text{ and } (2,2)$$

Some of the students find this not so easy and they can put their models to good use. It takes some a while to see that what counts is the distance  $r$  of the point from the origin, and that the height of the cone is then  $4 - r$ .

<https://www.desmos.com/calculator/btlqck2svx>

From here we go on to examine the contour diagram of the cone—a set of circles centred at the origin.

<https://www.desmos.com/calculator/xc5exgfpnt>

Now we turn our attention to the ant who is wandering around on the cone. We keep track of the journey with its path, its projection onto the base of the cone. Our main question is to calculate the ant's point of highest altitude.

For example, consider the path  $y = 4 - x$ . The students will find different ways to argue that the maximum point is  $(2, 2)$ . Perfect for grade 9.

The ant's journey in 3-space is the curve where the plane  $x + y = 4$  intersects the cone. Is it a parabola? 😊

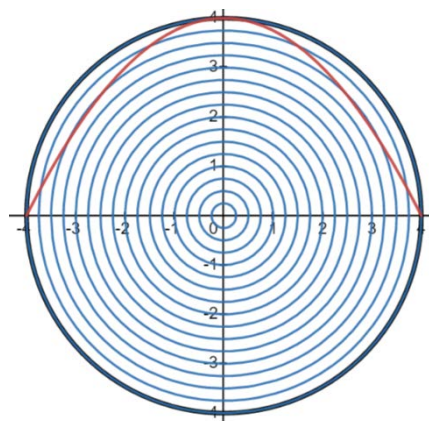
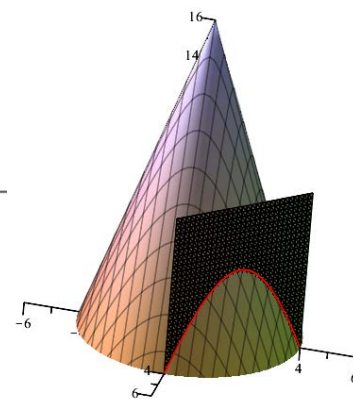
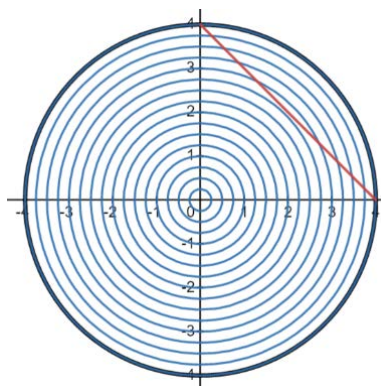
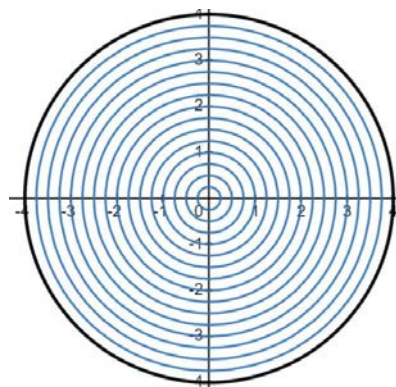
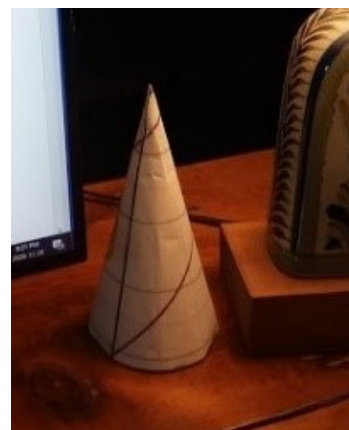
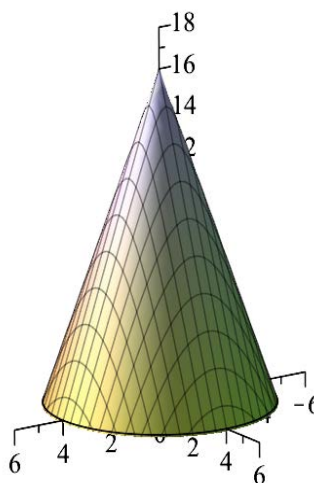
We look at other more interesting paths. Here's one. It's a parabola:

$$y = 4 - x^2/4.$$

Where is the maximum altitude?

This is a great grade 12 problem, but it does not require calculus. To minimizing  $x^2 + y^2$  with  $y$  set equal to  $4 - x^2/4$ , we get a quartic polynomial but it is quadratic in  $x^2$ , and can be handled with algebra.

[Ant Cone Teaching Manual](#)



## Beanstalk Scholarship-- a strange tale

The beanstalk is now strong enough to climb. Jacqueline puts on her jacq-boots, packs her jacq-cheese sandwich, takes up her trusty jacq-knife, and begins her ascent. However, even as she climbs, the beanstalk continues to grow. How fast is she climbing? How fast is it growing? The problem to be solved is this--will she ever get to the top, and if so, how long will it take?

Here's the data. The beanstalk grows at the rate of 2% per minute. To be more precise, this growth is uniform along its length in the sense that every vertical section of the stalk increases in length at that rate. So that, for example, when Jacqueline is 50 m from the top, that 50 m length above her will increase by 2% in the next minute.

Jacqueline climbs at the rate of 5 m/min, and at the moment she starts climbing ( $t = 0$ ), the beanstalk is 200 m tall.

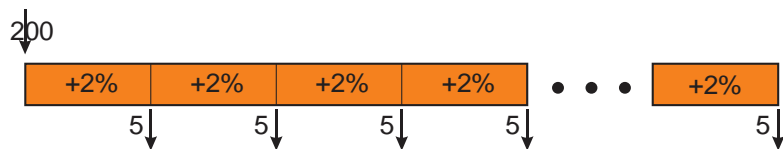
What makes the problem complicated is that both factors--Jacqueline and the stalk--are changing continuously and for that case we need the tools of calculus to make progress. However here we are going to study a discrete version by separating the two factors out.

*The discrete version.* Suppose that each 1-minute interval of Jacqueline's journey has two parts. In the first 30 seconds the beanstalk grows by 2%, and Jacqueline takes a small rest (and perhaps even gets a chomp on her jacq-cheese sandwich). In the second 30 seconds the beanstalk doesn't grow and Jacqueline climbs 5 meters.

What we have here is a discrete-time dynamical system with both additive and multiplicative growth. There are standard ways of handling this type of problem, often covered in a university course on dynamical systems.

Imagine our surprise upon realizing that this was just a disguised version of the standard scholarship problem often studied in Grade 11 financial math.

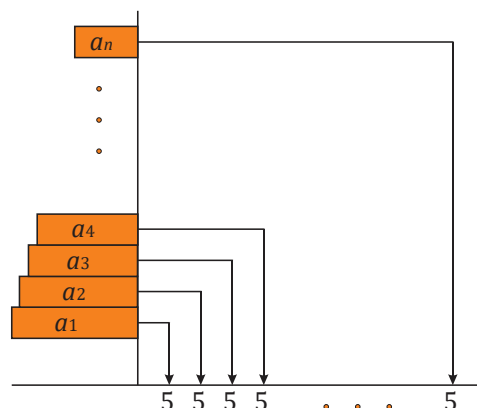
Dollars is replaced by metres. Jacqueline starts with a capital fund of 200 metres and it grows at an "interest rate" 2%. Now at the end of each 1-minute interval, She "withdraws" 5 metres by climbing. The question is "will she reach the top?", that is, will the capital fund run out and if so, when?



These regular 5 m payments are funded by the original 200 m investment, but later payments need less of that 200 m capital as they have more time to grow at the 2% rate.

That approach gives us quite an elegant solution to the beanstalk problem.

[Beanstalk Teaching Manual](#)





## Conditional Probability

### Wonder Cards

Wonder cards are either red or blue, and each card is either happy or sad.

- 40% of all cards are red.
- 60% of all cards are blue.
- 3/4 of all red cards are happy.
- 1/3 of all blue cards are happy.

I choose a random card and it turns out to be happy. What is the probability that it is red?

This is a simple problem which we use to explore conditional probability. The data allow us to produce the table at the right (for a deck of 100 cards). From this we ask the students to find:

$P(R|H)$  the probability that a happy card is red.

Of course what we are given is the “reverse”:

$P(H|R)$  the probability that a red card is happy

This often happens in probability: you are given one conditional probability and you need to find the reverse. Of course, that’s what Bayes Formula is all about.

$$P(R|H) = \frac{P(H|R)P(R)}{P(H)} = \frac{(3/4)(0.40)}{0.50} = 0.60$$

### [Wonder Cards Teacher Manual](#)

### School Coffee

Students in their spare period often visit a coffee house that is close to the school. During one period the manager kept track of coffee sales to students and reported that there were twice as many girls as boys. But when the results of the survey were reported in the student paper, the headline read: “Girls Twice as Likely as Boys to Drink Coffee.”

Are these two assertions equivalent? [Note that we assume all coffee drinkers buy their coffee at the coffee house.]

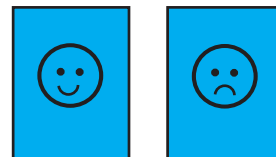
This is a case in which the two conditional probabilities are confused. To give the students a concrete problem, we give them the school diagrammed at the right with 720 students, 400 girls and 320 boys. Are either of the two assertions true for this school?

We then ask the students for a general analysis. For example, find general conditions under which the following two statements are equivalent.

- coffee drinkers are twice as likely to be girls as boys.
- girls are twice as likely as boys to be coffee drinkers.

### [School Coffee Teacher Manual](#)

Here we provide a set of four activities that give us an interesting journey through this fundamental aspect of probability.



	😊	☹️	
Red	30	10	40
Blue	20	40	60
	50	50	



	C	NC	
G	50	350	400
B	20	300	320

## *Sleeping Beauty*

Sleeping Beauty has an unusual neurological condition. When she wakes in the morning she cannot recall events of the past few days, and it takes an hour for her short-term memory to return.

One thing she does remember is that she is in the middle of an ongoing experiment. Every Sunday evening a fair coin is flipped and if it is heads, she is awakened Monday morning at 7, and if it is tails, she is awakened at 7 both Monday and Tuesday mornings. Otherwise (on all other mornings) she sleeps peacefully till 8.

One morning she is wakened at 7, and asked for the probability that the coin that was flipped the previous Sunday was heads. Just to be clear about her memory, at that moment, she knows that she has been awakened but she does not know whether it is Monday or Tuesday.

Since she studied probability with RabbitMath, she knows how to work this out. What answer does she give?

[Sleeping Beauty Teacher Manual](#)



## *Bicycle Helmet*

Consider the following “advertisement” for bicycle helmets (which actually appeared in a magazine).

***80 cyclists died this year in car/bicycle collisions.  
Only 24 of these were wearing helmets.  
Always wear a helmet when cycling!***

(a) Discuss this ad. What do you think its purpose is? What conclusion does it want you to draw? Do the data it presents justify this conclusion? Your explanation should use the ideas of conditional probability but be accessible and informative to someone who had not formally encountered those ideas.

(b) Present what you think would be a better ad, manufacturing additional data if you need to. But remember—effective ads need to be as simple and direct as possible.



[Bicycle Helmet Teacher Manual](#)

## Dart board

Suppose you are throwing darts at a dart board and on each throw, you have probability  $1/20$  of hitting the bulls-eye. How many darts do you have to throw *on average* to get the first bulls-eye?

The answer is 20.

*The Dartboard Theorem.* If on each throw, you have probability  $p$  of hitting the bulls-eye, then the average number of throws to get the first bulls-eye is  $1/p$ .

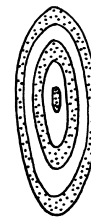
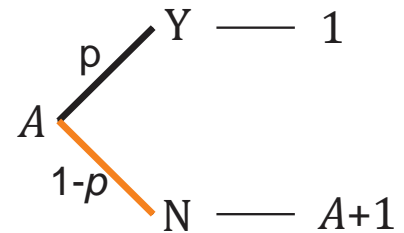
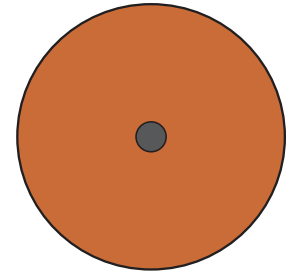
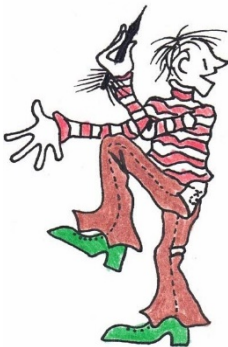
The argument is exactly the one we found what we were rolling a die to get the first 6.

$$A = p(1) + (1 - p)(A + 1)$$

and this solves to give

$$A = \frac{1}{p}$$

It's remarkable how often this principle arises in different problems. I find this dartboard model simple and general enough that it can be readily recognized in a number of contexts.



And now as a challenging extension of this recursive approach, here's a fascinating pair of examples.

- I roll a fair 6-sided die until I get a six followed by a five.  
How many rolls on average does that take?
- I roll a fair 6-sided die until I get two sixes in a row.  
How many rolls on average does that take?

We have rolled this result into the **Dice Roll** activity.

[Dart Board Teaching Manual](#)



## Desmos Suite

We make plentiful use of Desmos in our activities and there are a number of reasons for that. The main one is that most teachers use it and are comfortable with it. Related to that students, even those who have done very little coding, catch on quickly.

Our main objective of course is that students enjoy constructing things, especially structures that are artistic and “neat,” and dynamic. We also find that the construction process is challenging in an interesting way, involves the student in working with structures and computational thinking.

The dynamic behaviour of these activities require that the students learn how to work with a parameter, something they seem to have little experience with. For example, lines for them are equations such as  $y = mx + b$  whereas these activities work with structures such as  $A + tB$  or  $A + t(B - A)$  where  $A$  and  $B$  are points and  $t$  is time. Once they get onto that, they catch fire.

### Parametric Lines

*Parametric Point.* The theme here is to understand how to draw lines parametrically, in particular to learn how to move a point along a line. We use Desmos as our tool here and it’s interesting to see what you can do working only with lines.

We work up slowly to the animation on the right in which, as we move  $P$  and  $Q$ , the line between them moves along with them.

In terms of the use and power of parameters, it is well worth studying (and playing with) the note on the use of the parameter  $t$  in the Desmos site cited above.

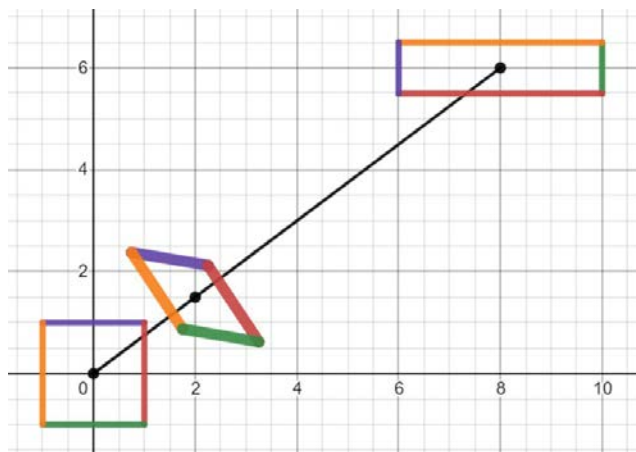
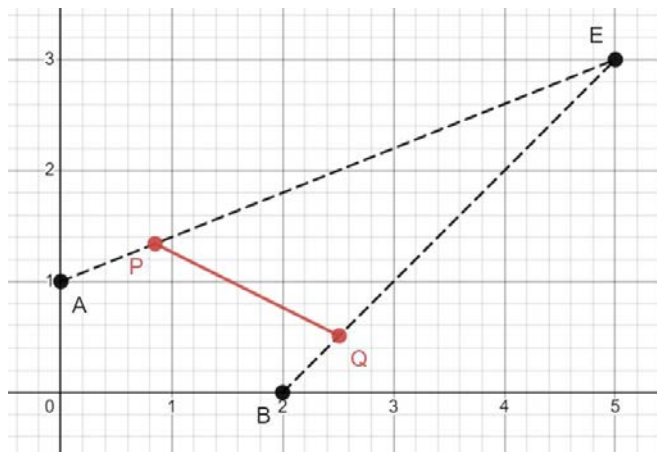
Having mastered that, we are ready to have some straight line fun.

<https://www.desmos.com/calculator/uhnyaxodul>

*Parametric Line.* Here we see a square seeming to rotate into a rectangle. How did we do that? Well it seems more complex than it really is—in truth, all points are simply moving in a straight line.

Since the starting and ending rectangles have different orientation, the image will have to make a flip. In that case, will there have to be a moment at which all four corners will lie in a straight line?

<https://www.desmos.com/calculator/2fvr1gw4ac>

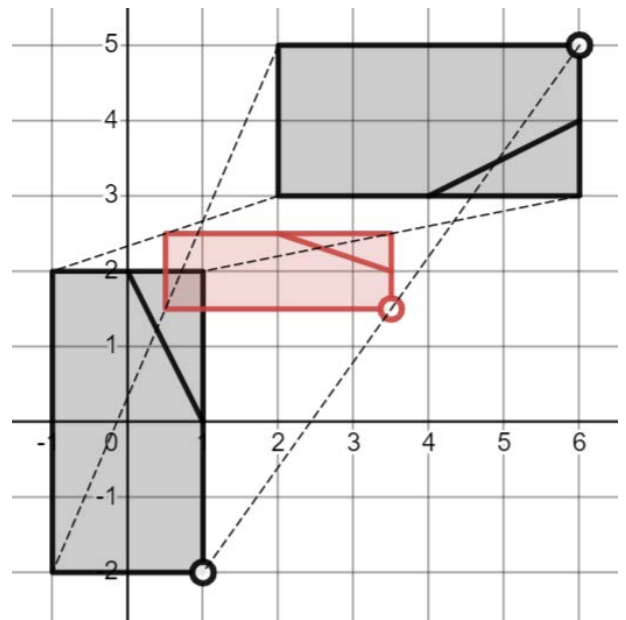


*Parametric Rectangle.* This is an interesting problem. It's perfect for grade 9, but it works well at many levels. I put it on the final exam (take-home: 48 hours long) in my linear algebra course and many students didn't know what to do—though we had done the general theory of linear transformations.

Over the time interval  $0 \leq t \leq 1$  the  $4 \times 2$  rectangle centred at the origin transforms into the  $2 \times 4$  rectangle centred at  $(4, 4)$ , such that every point  $(x, y)$  moves in a straight line at constant speed to its corresponding destination.

(a) At one point in the journey of the rectangle, it becomes a line segment. At what time does this happen? On the diagram, make a careful sketch of the line.

(b) At one point in the journey of the rectangle it becomes a square. At what time does this happen? On the diagram, make a careful sketch of the square.  
<https://www.desmos.com/calculator/yt53a9m1qt>



*Parametric Triangle.* Here's another "straight-line" transformation. This time we use a triangle. The points A, B and C all travel in a straight line.

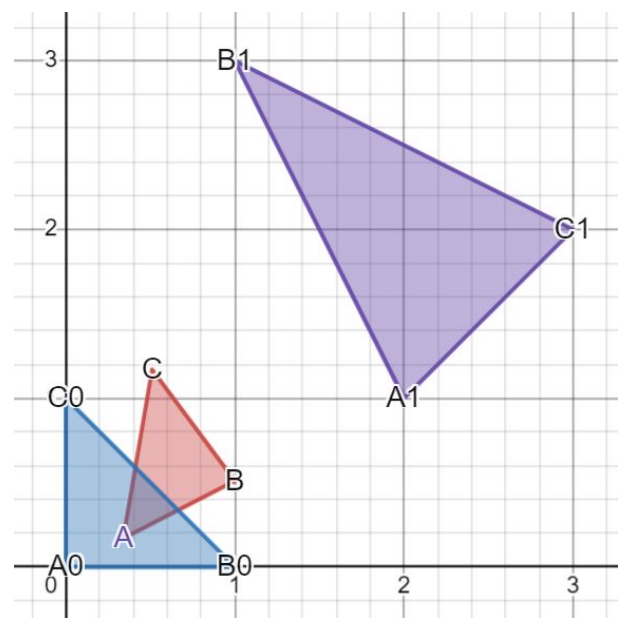
But there are some interesting questions.

(a) There is a change in orientation, and that means that at some time the moving triangle will have to be a straight line? When does that happen?

(b) The blue and purple triangles are both isosceles. Is there a time at which the pink triangle is also isosceles? The answer is quite unexpected.

For these questions, the students need the quadratic formula. We prefer to have that wonderful tool delivered in grade 9, but grade 10 would be a wonderful place for this activity.

<https://www.desmos.com/calculator/zvtdvxcwuo>



[Parametric Lines Teacher Manual](#)

## Rotate!

*SinCos.* Naturally enough, our grade 9 students want to incorporate rotations into the life of the lines they are working with. Well trig functions don't come till grade 10, and in fact "rotations" arguably don't appear till grade 11. Never mind, the sine and cosine functions can simply be treated as useful apps. The students are quite happy with that game.

Quite simply if I take the point  $(1, 0)$  on the  $x$ -axis and rotate it through  $50^\circ$  about the origin, the resulting point has coordinates  $(\cos(50), \sin(50))$ . We can even take that as the definition of  $\cos$  and  $\sin$ . Then if we replace  $50$  by  $T$ , the point rotates around the unit circle.

$$A = (\cos(T), \sin(T))$$

<https://www.desmos.com/calculator/vgtkcp1b4b>

Now we take that point  $A$  and make it one end of a rotating stick. The other end  $B$  provides a new, interesting concept, that of a phase shift. Whereas  $A$  starts on the positive  $x$ -axis (angle  $0$ ),  $B$  starts on the negative  $x$ -axis (angle  $180$ )

$$B = (\cos(180 + T), \sin(180 + T))$$

<https://www.desmos.com/calculator/sorfqi4wz8>

And then, working in small groups, we challenge them to adapt and extend these simple trig expressions to more complex structures.

*Spiral stick.* We can use this example to show them two new behaviours. The first is how to move the centre of rotation. Again, if they are in grade 9 we give them the equation  $y = p(x)$  of the parabola and they seem to quickly figure out how to move the centre of the stick along it. Note that the rotation here is counter-clockwise.

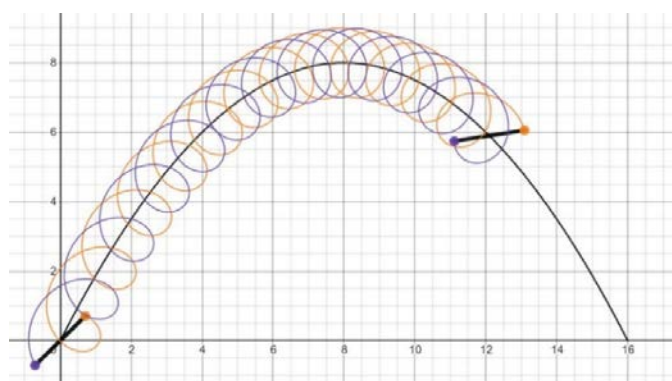
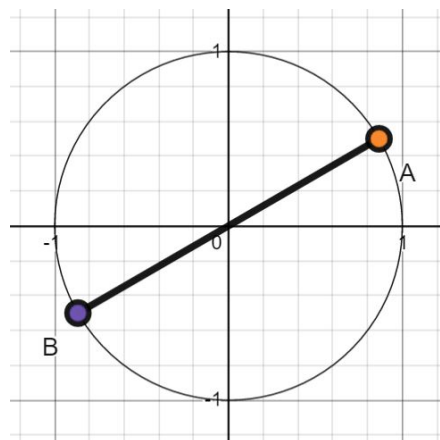
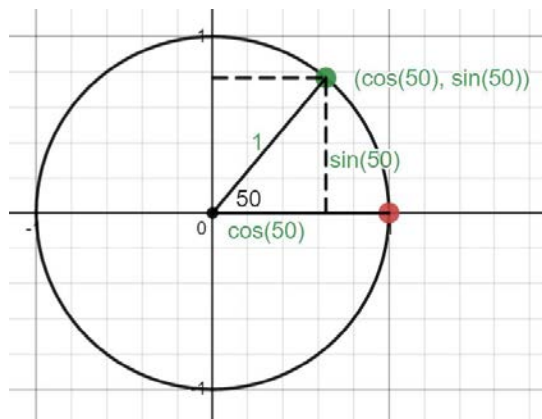
$$A = (T + \cos(45 - 360T), p(T) + \sin(45 - 360T))$$

The second thing we do is show them how to "paint" the trajectory of the particle. For  $\{0 \leq t \leq T\}$ . This paints the spiral "history" of the endpoints  $A$  and  $B$ .

$$A_1 = (t + \cos(45 - 360t), p(t) + \sin(45 - 360t))$$

Play with this and see.

<https://www.desmos.com/calculator/6adsnodwvl>



*Emerging Name*

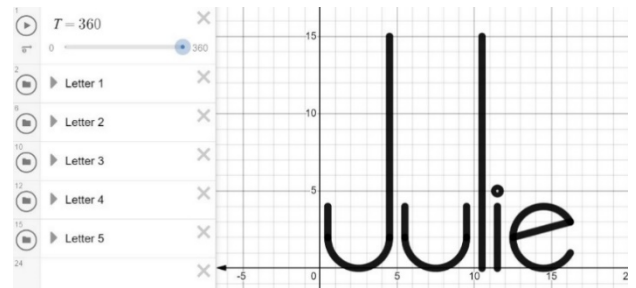
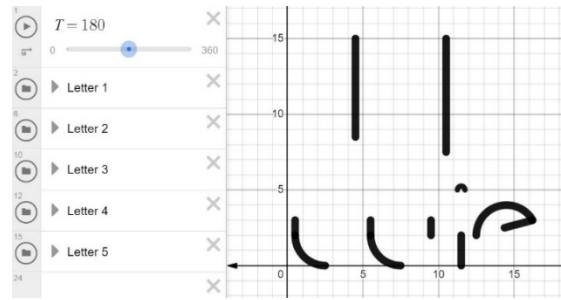
<https://www.desmos.com/calculator/w1trka7hqi>

As the slider slides, a name begins to appear.  
Who can be the first to guess the name?

Why it's Julie—one of the counsellors!

And then of course everyone starts to build their own name.

This is a perfect follow-up to our new-found ability to draw circles.

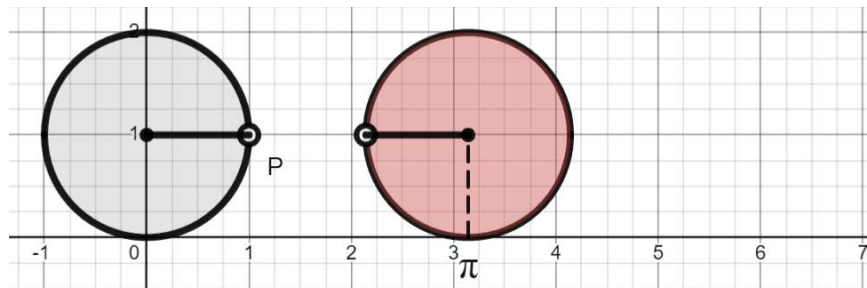


*Wheel along Axis*

<https://www.desmos.com/calculator/4q35i2slfx>

We begin with a wheel with centre  $(0, 1)$  and radius 1, tangent to the  $x$ -axis at the origin. Suppose that, starting at time  $t=0$ , the wheel rolls along the  $x$ -axis to the right, with no slipping, at a speed of  $\pi$  units per second.

On the diagram below, we sketch the wheel at times  $T = 0$  and  $T = 1$ .



The problem is to animate the wheel, plotting student the coordinates  $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$  of the centre of the wheel, and  $\begin{bmatrix} x_P \\ y_P \end{bmatrix}$  of the marker point at any time  $T$ .

### Wheel around circle

<https://www.desmos.com/calculator/tybplls9cr>

This time we are going to roll the wheel, without slipping, around the circle

$$x^2 + y^2 = 4$$

of radius 2. Starting with the centre of the wheel at (3, 0), the wheel moves upwards causing it to rotate about its centre in a positive (counter-clockwise) direction.

As the wheel rolls, its centre follows a circle of radius 3:

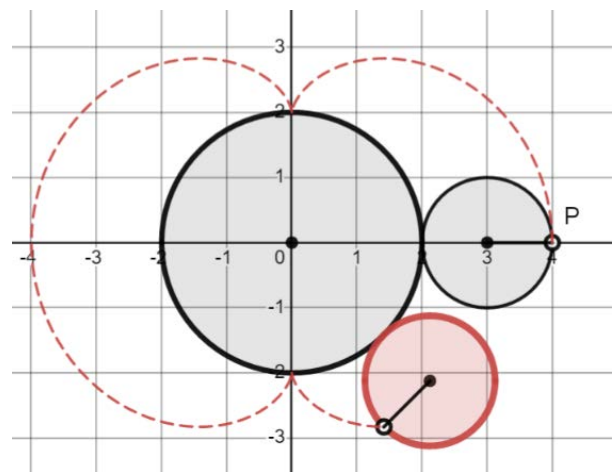
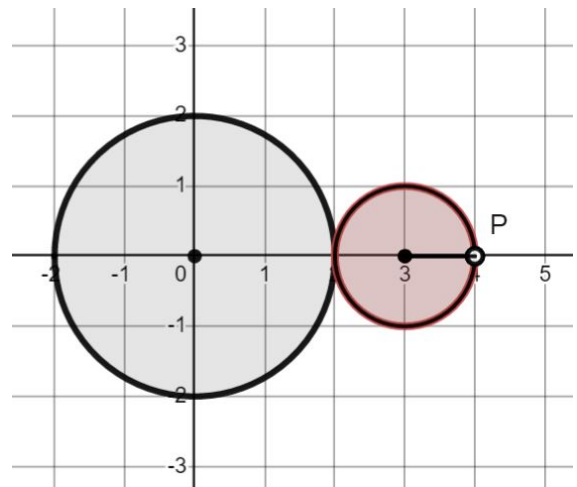
$$x^2 + y^2 = 9$$

We measure time  $T$  so that the wheel revolves around circle at the rate of  $90^\circ/s$ , such that in  $T$  seconds, the wheel revolves through an angle of  $90T$ . Thus, at time  $T = 1$ , the wheel is on top of the circle.

The problem in building the animation, is to figure out how fast the wheel *rotates* about its centre while it is *revolving* around the origin. We know the rate of *revolution* ( $90^\circ/s$ ) but what is its rate of *rotation* (about its centre)? That simple question turns out to be more interesting than it first appears.

When we give them a problem that changes the size of the circle or the wheel, we find that some students want to come up with a general relationship: given the ratio between the size of the wheel and the size of the circle, what is the resulting ratio between the speed of rotation of the wheel and the speed of its revolution about the origin.

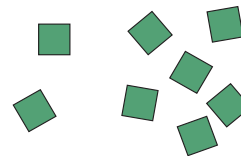
[Rotate! Teacher Manual](#)





## Dice game

Here's the game. You can choose any number  $n$  of dice that you want and the dice are rolled. If any of the dice show a 1, you get zero. Otherwise (if no 1) you get the sum of the outcomes.



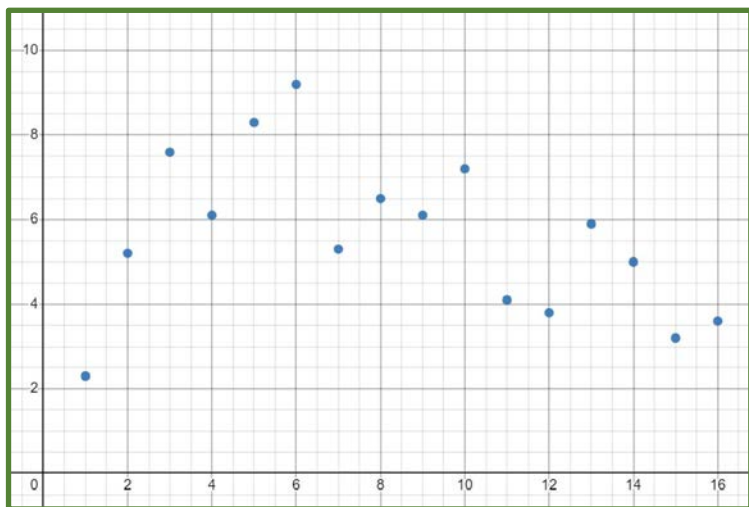
How many dice should you choose to maximize your average payoff?

First of all we must note that once you have chosen the number of dice to use (for example, if you choose 3 dice), you will get lots of different outcomes. Every time you get a 1, you will get payoff zero, and otherwise you'll get the sum of the dice which will be at least 8. Anyway, lots of different numbers. So what we want to do is take the average of all those possibilities.

In this way we will get a different average for each value of  $n$ . And we will want to choose the value of  $n$  with the highest average.

*Of course we start by generating some data.*

We get out our box of 100 dice and have the students play the game many times with different numbers  $n$  of dice from 1 to 16. The average payoffs they got for each fixed value of  $n$  are plotted below.



$n$	Average payoff
1	2.3
2	5.2
3	7.6
4	6.1
5	8.3
6	9.2
7	5.3
8	6.5
9	6.1
10	7.2
11	4.1
12	3.8
13	5.9
14	5.0
15	3.2
16	3.6

<https://www.desmos.com/calculator/kvgml9bco9>

Is this pattern of data what you might expect? The answer is yes—a tradeoff between too few dice and too many.

With a grade 8 understanding of probability, and the hugely important idea of a weighted average, the kids can work out the expected score for any number  $n$  of dice and can use Desmos to plot the results on top of their empirical data.

[Dice Game Teaching Manual](#)

## Dice roll

I roll a fair 6-sided die until I get my first 6. How many rolls on average does that take?

### Collecting some data

Most teachers have a box of dice in the classroom and the students can use these to generate lots of data.

At the right we tabulate the results obtained from 40 trials. Of these trials,

4 got the first 6 on the 1<sup>st</sup> roll,

6 got it on the 2<sup>nd</sup> roll,

3 got it on the 3<sup>rd</sup> roll,

etc.

The longest number of rolls needed was 12.

<https://www.desmos.com/calculator/oqirrn18zr>

What we are interested in is the *average* number of rolls needed. The data we have collected will give us an “empirical” estimate of the average  $A$ . The estimate we get is

$$A = \frac{4(1) + 6(2) + 3(3) + 4(4) + \dots + 0(11) + 1(12)}{40} = \frac{182}{40} \approx 4.55$$

Notice the structure of this expression. It is the average of 40 numbers, 4 of which are 1, 6 of which are 2, 3 of which are 3, etc. We add those 40 numbers together and then, to make this an average, we divide by 40. The result we get is  $A = 4.55$

Okay:

This quantity 4.55 is an estimate of some theoretical measure of the average, and that’s what we would like to calculate.

Now there are a number of quite different ways to do that. The one we will work with here is my favorite. It is arguably the simplest and the most powerful. It uses that wonderful tool: recursive thinking.

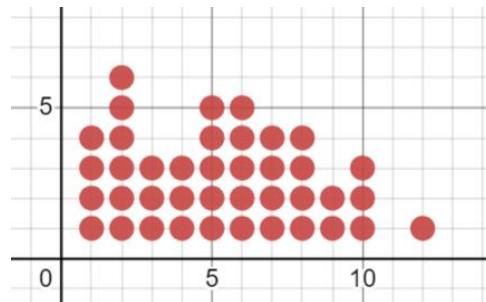
Take your first roll. With probability  $1/6$  you’ll get a six and you got it in one roll. With probability  $5/6$  you will *not* get a six and then what?—well then you can expect to get  $A$  *more* rolls for a total of  $A + 1$ . We formulate this as a recursive equation:

$$A = \frac{1}{6}(1) + \frac{5}{6}(A + 1)$$

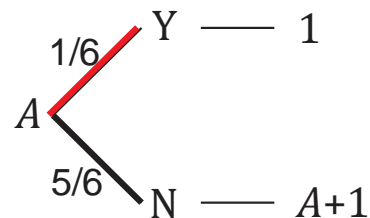
and this solves to give  $A = 6$ .

It’s interesting that it takes 6 rolls on average to get the first six. What’s going on there?

[Dice Roll Teaching Manual](#)



# rolls	# trials
1	4
2	6
3	3
4	3
5	5
6	5
7	4
8	4
9	2
10	3
11	0
12	1
total	40



*What’s going on here?*  
 What going on here is what I have christened the *Dartboard Theorem*. Find out what it’s all about.

## Drive

<https://www.desmos.com/calculator/bzm3elabej>

How fast do you drive to minimize fuel consumption?  
To tackle that question we need to know how fuel consumption changes with speed.

The graph at the right is derived from a simple physical model that take account of the basic factors at work: friction, air resistance and engine efficiency. We conclude this unit with a discussion of this model. That should work well for grade 12 students, particularly those taking the Physics course.

But our first approach to this problem is graphical and is perfect for grade 11.

The given graph plots litres/hour against speed, whereas to minimize cost for a trip of a given distance we want litres/km. So we need to understand how to interpret that on the graph.

It turns out that while L/h is a “height” on the graph, L/km is a “slope.” The gas consumption per km at speed  $v$  is the slope of the secant drawn from the origin to the corresponding point on the graph. For example, if you drive at 80 km/h, you will use close to  $4/80 = 0.05$  litres for each kilometer.

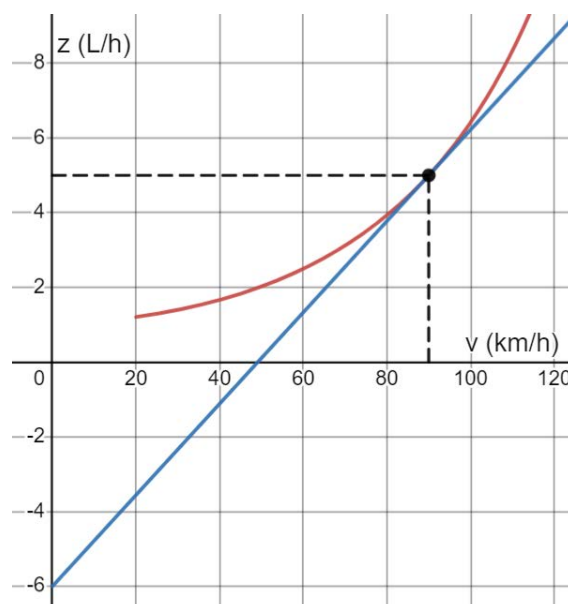
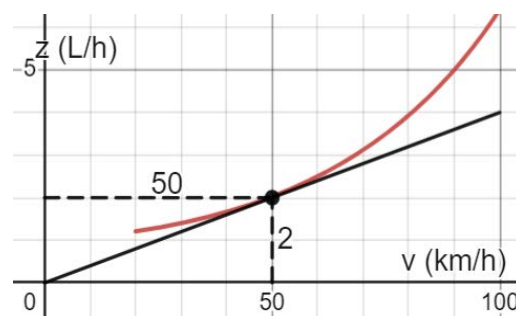
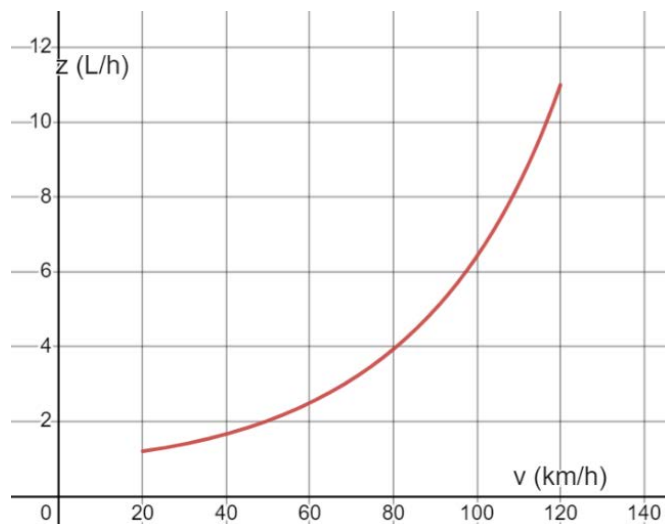
Thus, the optimal speed is where the slope of the secant is minimized and that happens at  $v = 50$  km/h.

But nobody drives that slowly. Does that mean that they don't care about climate change? Well it's more likely, because they have put a value on their time.

We build a simple model for that scenario, and the optimum is attained where a line drawn to the curve from a point 6 units *below* the origin has a minimum slope, and that happens at  $v = 90$ .

*A remark.* The graph at the right belongs to a fuel cost of \$2/L and a time valuation of \$12/h. That 6 in the diagram is actually the ratio  $12/2$ .

[Drive Teaching Manual](#)





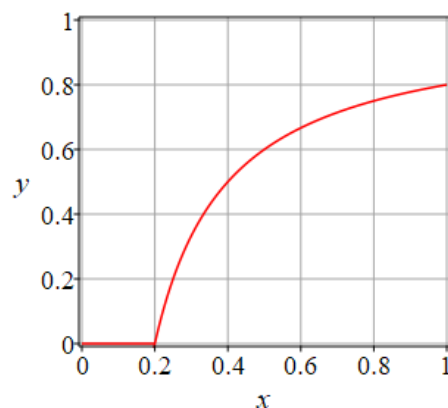
## Greedy

### *The investment*

To start with you choose any number  $x$  between 0 and 1. We'll call  $x$  your "investment." There is a reason your investment should be large—the higher the investment, the higher will be the payoff each time you play the game. But there is also a reason your investment should be small—the higher the investment, the more likely you are to have to leave the games after each round. Somewhere in the middle there ought to be an optimal investment.

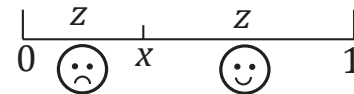
### *The payoff.*

Each time you play the game you'll get a payoff  $y = f(x)$ , whose graph appears at the right. If you invest less than 0.2 the payoff is zero. For  $x > 0.2$ , the payoff increases with  $x$  following the graph. For example, If you invest  $x = 0.4$ , you will get a payoff of  $y = 0.5$  with each play of the game.



### *The z-effect*

After each play of the game, a random number  $z$  is generated between 0 and 1. If  $z$  is greater than your investment  $x$ , you get to play again and get another payoff of  $f(x)$ . Following that, another random number  $z$  between 0 and 1 is generated and if it is greater than  $x$ , you play again, etc. Your winnings accumulate over successive plays.



The game goes on like this until the random number  $z$  is less than  $x$ . At that point the game is over and you take your total winnings home.

The question of course is to find the value of  $x$  that will maximize your expected winnings.

Interestingly enough, early in the analysis, we run into the dartboard theorem!

### *Two solutions: graphical and analytical.*

The graphical solution features the familiar construction of drawing secants from the origin to the graph.

The analytical solution starts with the formula for the graph:

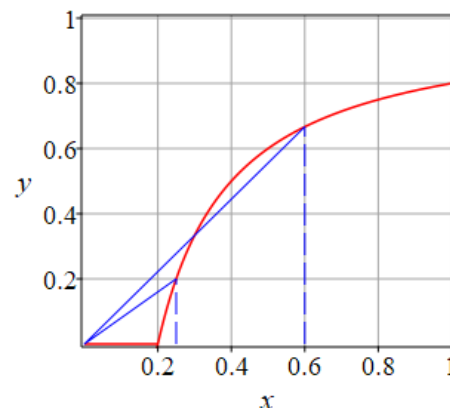
$$f(x) = 1 - \frac{1}{5x} \quad \{0.2 \leq x \leq 1\}$$

and then turns to, well, maybe calculus. That certainly works for grade 12.

But in fact there is also a grade 11 solution using nothing more than being able to find the vertex of a parabola.

A nice problem.

[Greedy Teaching Manual](#)





## Guinea Egg

To cook a guinea egg you need to raise its temperature to  $T = 90^\circ$ , but guinea eggs are delicate so this must be done slowly. There are two established methods.

The first is to put it in boiling water at a constant temperature of  $100^\circ$ . The second is to put it in a microwave oven at low power.

The surprising thing is that if we start with the egg at room temperature ( $T = 20^\circ$ ) the two methods have exactly the same cooking time--12 minutes.

Nevertheless, the two methods employ different forms of heat transfer and this affects the shape of the temperature-time graphs, as illustrated at the right. What's going on here? This is an excellent question for the class to discuss.

<https://www.desmos.com/calculator/iqrhq3lskh>

Here we look at the question of how to shorten the cooking time by moving the egg from one option to the other. Any scheme that the students come up with have to be illustrated on the graph and that's a nice challenge for some students.

Finally a wonderful surprise. The water graph follows Newton's Law of temperature change. This is nicely illustrated in the diagram at the right and a good problem for any student who has worked with exponential change, is to use the diagram to find the equation of the curve.

<https://www.desmos.com/calculator/2v6gq7ydne>

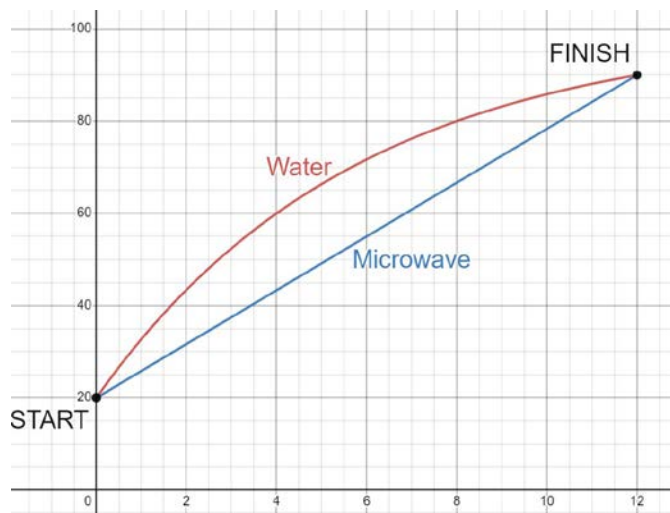
What the graph tells us is that the temperature difference

$$D = 100 - T$$

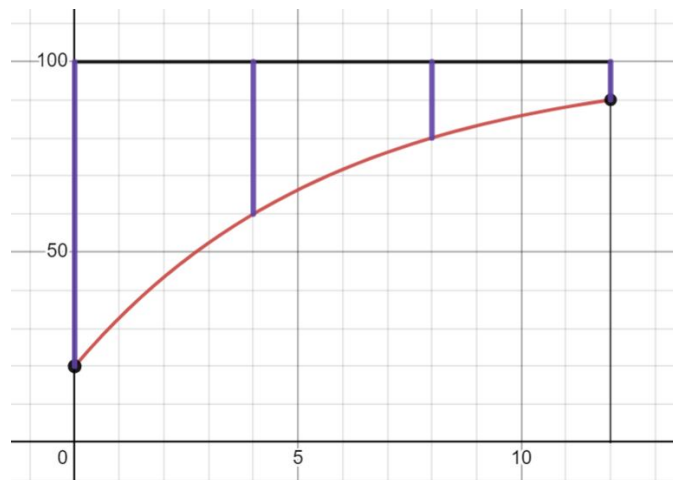
between the water and the egg is cut in half every 4 minutes. And it starts (at  $T = 0$ ) at  $80^\circ$ . Is that not all you need to find the equation?

Of course a grade 12 calculus student can then use the equation to check the optimal switch point.

[Guinea Egg Teaching Manual](#)



This is an excellent grade 9 activity. The discussions of slope and concavity are right in line with the basic ideas of the course as are the graphical interpretation of moving the egg from one option to the other.



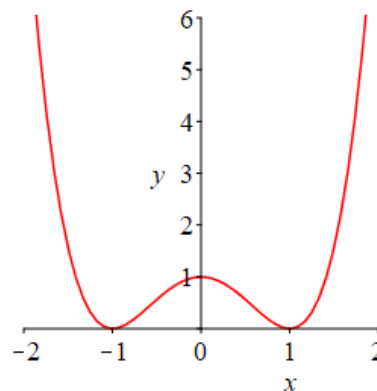
## Inverse Quartic

An important notion in Grade 12 is that of the inverse of a function, but the examples that the students are given to work with are often not so interesting. Understanding comes from examples that have a rich enough structure. I have been happy with the level of complexity that this example provides.

The graph of the quartic (degree 4) polynomial

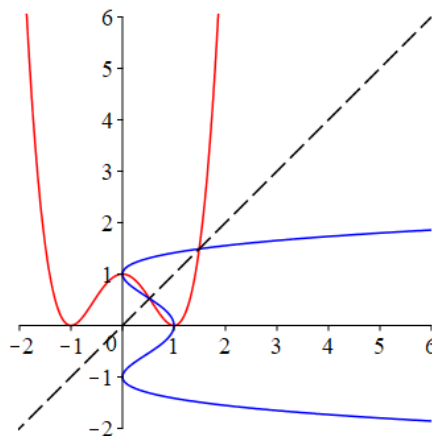
$$F(x) = (x - 1)^2(x + 1)^2$$

is drawn at the right. Take note of the double roots at  $x = 1$  and  $x = -1$ , and of the symmetry about the  $y$ -axis.



The inverse of this graph is drawn at the right. It is the reflection of the original in the line  $y = x$ . Points  $(a, b)$  in the original graph correspond to points  $(b, a)$  in the inverse.

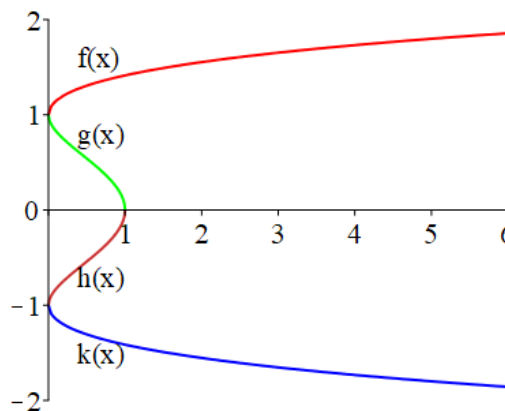
Although  $F(x)$  is a function, its inverse is not. For a graph to be a function, each value of  $x$  must give a unique value of  $y$ . Geometrically, that means that each vertical line can intersect the graph at most once. For the graph of the inverse, values of  $x$  between 0 and 1 give *four* corresponding values of  $y$ . Two of these values are positive, while the other two are negative. Since the graph is symmetrical about the  $x$ -axis, the negative values are the opposites of the positive values.



However, if we break the graph into the four pieces shown in color at the right, each individual piece is the graph of a function. We can call these four pieces the “function pieces” and they are labeled from top to bottom as  $f(x)$ ,  $g(x)$ ,  $h(x)$ , and  $k(x)$ .

Your job is to find equations for each of these four functional pieces. In each case, pay attention to the domain of the function and check that in each case, the equation and the graph give us the same answer.

[Inverse Quartic Teaching Manual](#)



## Least Squares

We start with a simple set of points and we want to find the “best-fit” straight-line. We use the standard regression approach which for any line calculates its “score” as the sum of squares of the vertical distances of the points from the line. For example, the line  $y = x + 4$  has score

$$SSQ = \sum_i [y_i - (x_i + 4)]^2 = 11$$

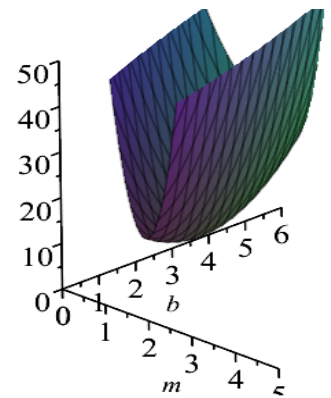
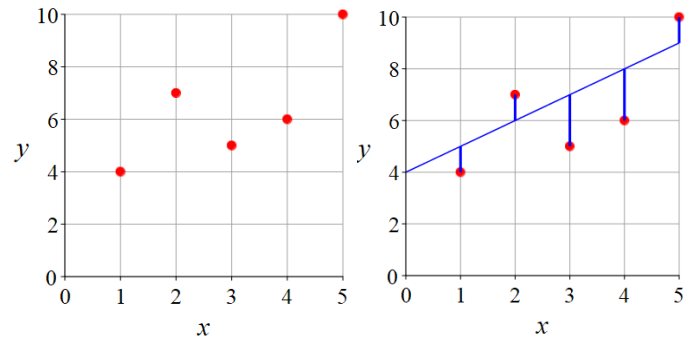
and the students can easily tabulate and calculate this. The “best-fit” line  $y = mx + b$  will be the one with the smallest score. Thus we want to minimize

$$SSQ(m, b) = \sum_i [y_i - (mx_i + b)]^2$$

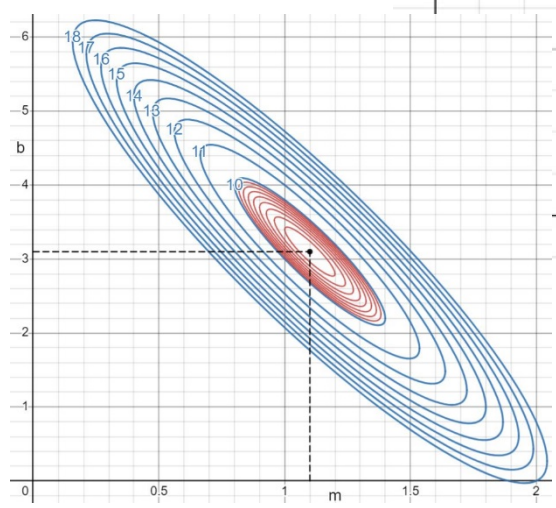
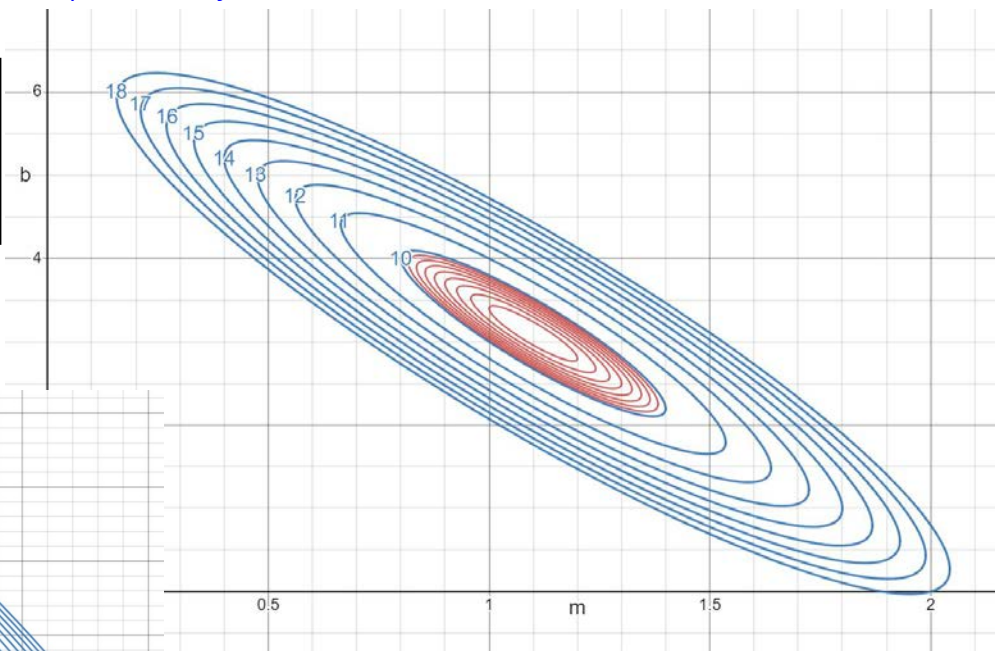
This is a function of two variables and we play with it in different ways. Its graph is a paraboloid sitting over  $m$ - $b$  space and we want to find its vertex. Here the vertical axis is the  $SSQ$  value.

How are we to work with a function of two variables? Its graph is a surface instead of a curve. Our answer will be to use the  $SSQ$  level curves.

<https://www.desmos.com/calculator/t1thfdw1dy>



Can you locate our starting line  
 $y = x + 4$   
 which has  $SSQ = 11$ ?



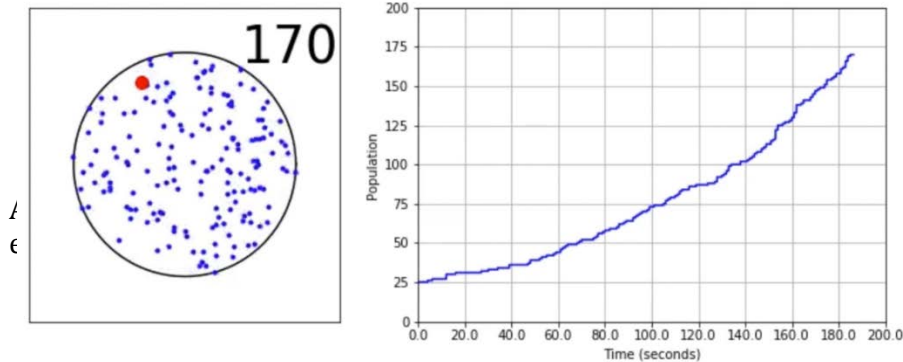
The blue contour lines run from a max of 18 to a min of 10 and then the red lines provide a finer gradation from 9.9 down to 9.2.

Desmos allows the students to play with this graph. They can locate the point  
 $(m, b) = (1.1, 3.1)$   
 where the  $SSQ$  is a minimum. And so much more...

[Least Squares Teaching Manual](#)

## Lightning

Here we have a population of particles bouncing around inside a circle. In addition there is a “lightning storm” with one strike every second at a random place inside the circle. Any particle that is within a fixed distance  $\delta$  of the strike produces an “offspring” that immediately become a new particle and goes off in a random direction. That’s how the population grows, and as it grows, its size is plotted. We have built a Python [animation](#) of the process and the students begin by watching it.



To put a scale on this process, we specify the following parameters:

radius of the circle:  $r = 5$  cm  
 lightning frequency 1 per sec  
 lightning proximity:  $\delta = 0.5$  cm  
 initial pop  $N = 25$

We ask the students whether this growth is exponential and if so, to find the theoretical equation for the population size.

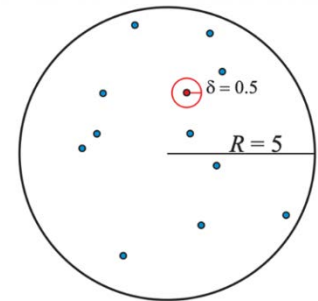
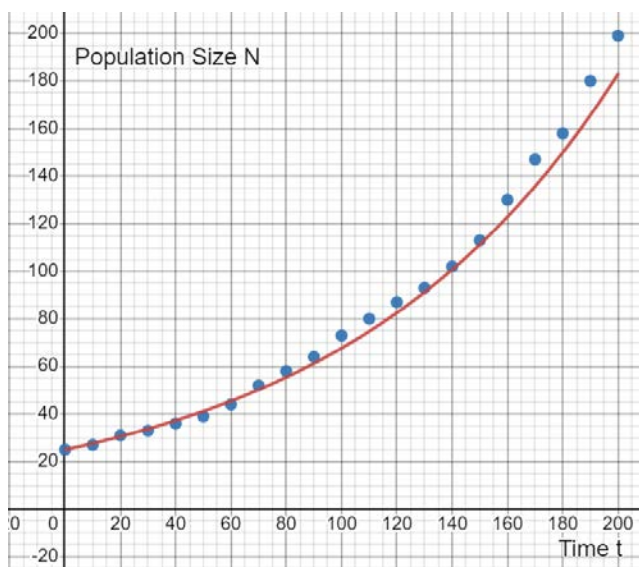
They assure us the growth is exponential but what we ask them why we find them unsure and even evasive. They say things like, the bigger the population, the faster it grows, but do not identify the essential mechanism that growth rate is proportional to size.

Indeed, the equation emerges from a careful probabilistic argument and arrives at the simple formula

$$N = 25(1.01)^t$$

We give the students the simulation data at 10-second intervals and they can use Desmos to check the fit to their theoretical model.

<https://www.desmos.com/calculator/lusgyw0t9c>



Time $t$	Size $N$
0	25
10	27
20	31
30	33
40	36
50	39
60	44
70	52
80	58
90	64
100	73
110	80
120	87
130	93
140	102
150	113
160	130
170	147
180	158
190	180
200	199

[Lightning Teaching Manual](#)

## ***m-b* Playground**

A straight line is determined by its slope  $m$  and its  $y$ -intercept  $b$ . This fact is enshrined in the formula

$$y = mx + b$$

which seems to be the students' number one choice for the equation of a line.

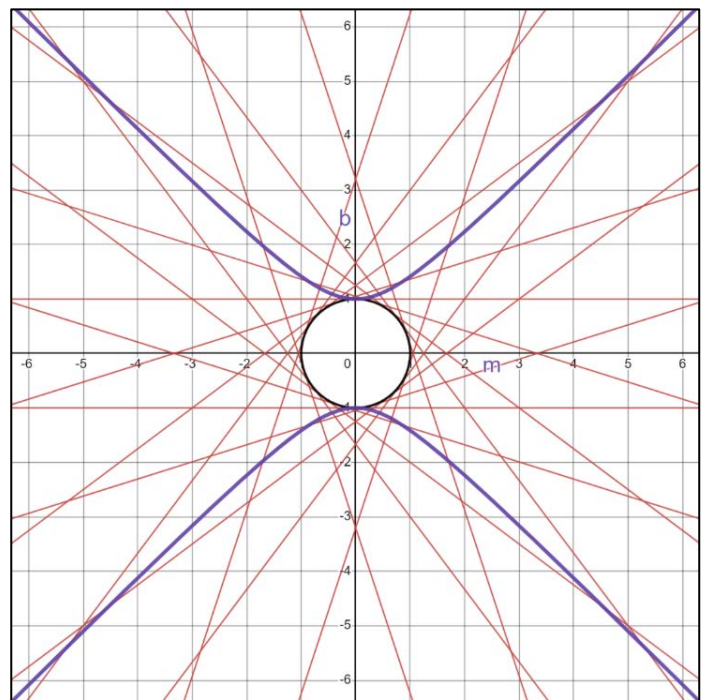
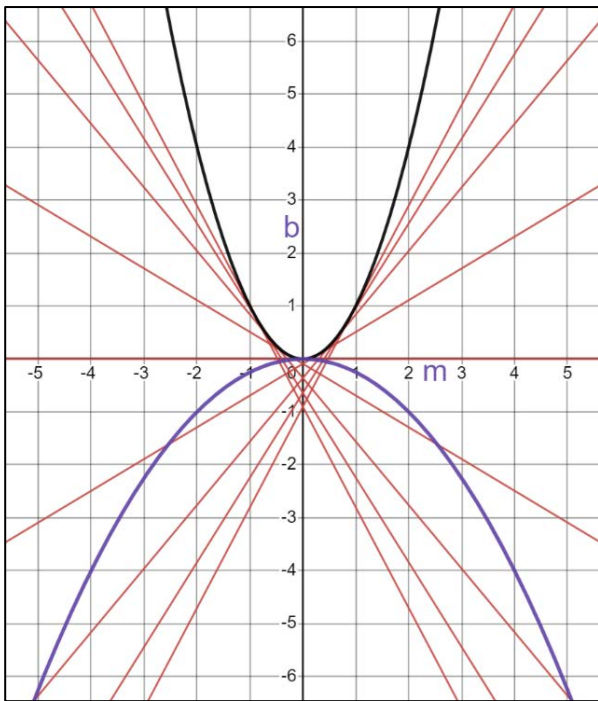
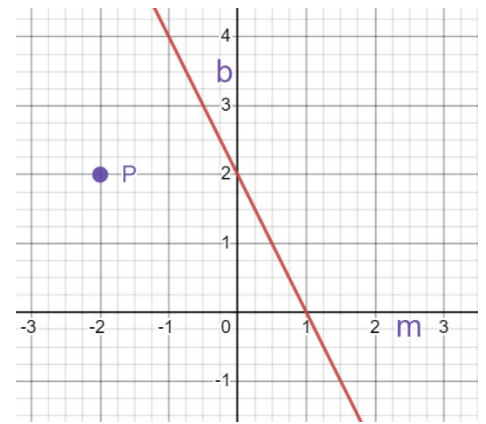
Another way to think of this is to say that a line in  $x$ - $y$  space corresponds to a point in  $m$ - $b$  space.

This is illustrated in the diagram at the right. Here the red line is drawn with respect to the standard  $x$ - $y$  axes but if we relabel the same axes as  $m$ - $b$  the line is identified as a purple point  $(-2, 2)$ .

We like this problem because it involves simple equations in the coordinate plane, but has that extra dimension of complexity.

If a line is a point, then an infinite family of lines (in  $x$ - $y$  space) is an infinite family of points (in  $m$ - $b$  space), which will typically be a curve or a line. That sounds like it might be a good playground.

We play with a few simple linear examples and then we look at a couple of beautiful problems-- the family of all tangents to the parabola  $y = x^2$  and the family of all tangents to the unit circle.

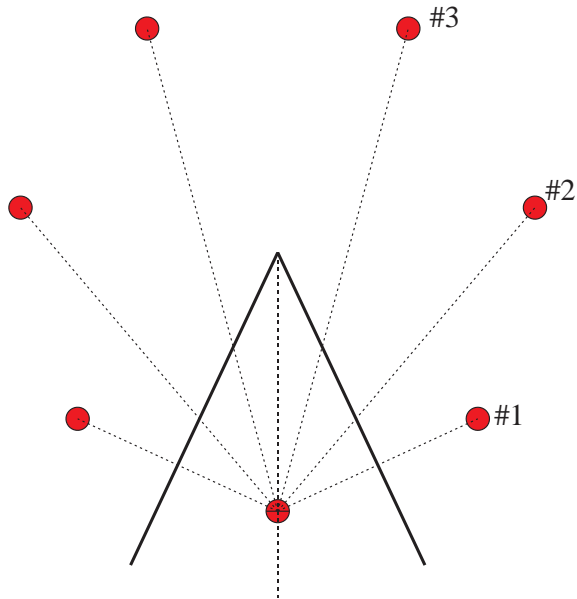
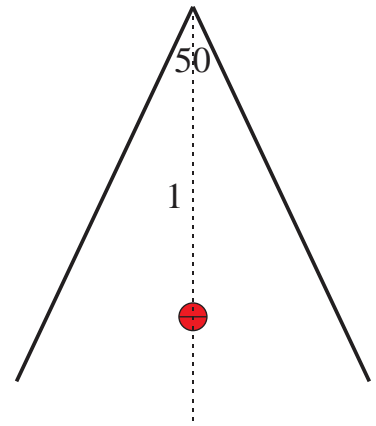




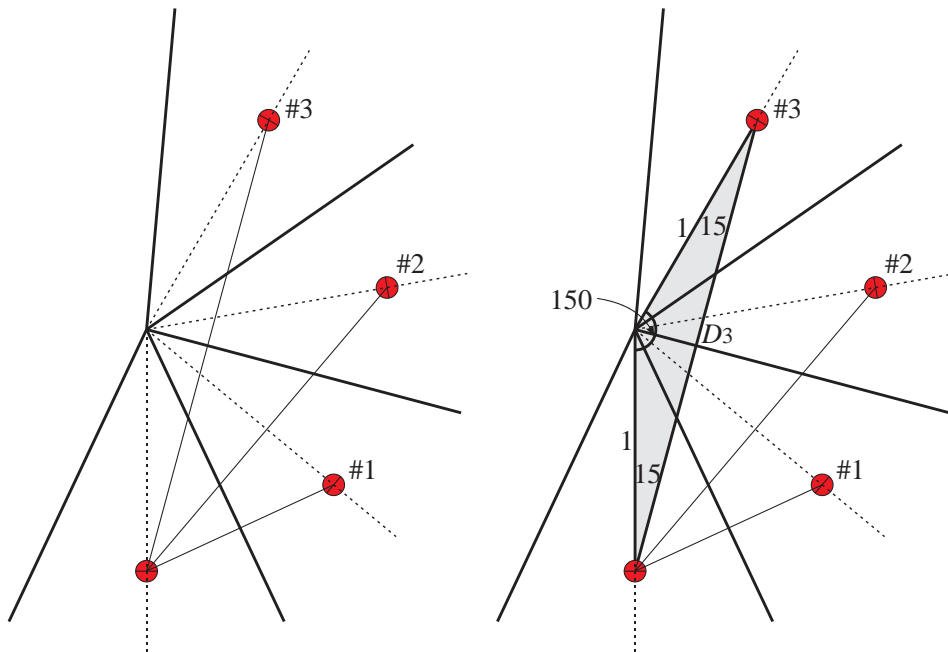
## Mirrors

You are in the middle of two mirrors set at an angle of  $50^\circ$  and your eyes are 1 meter from the vertex. You are looking in the mirror at your nose and you can see 6 images. Here's the problem:

- Plot, for each image, the path of the light ray as it travels from your nose back to your eye.
- Calculate, for each image, its apparent distance from you.



This is a great problem for small-group activity as there are so many ways to tackle this problem and every team will have a different solution. The analysis is a nice combination of trig and geometry and initial solutions are guaranteed to be somewhat disorganized. However, with a good measure of persistent play, a solution emerges that is elegant and even extraordinary.



## Missteaks

Here we take a fun look at arithmetic operation that are not supposed to hold.

But sometimes, with the right numbers, they do! In each case, the students are challenged to find other examples of each kind.

In the first one we are essentially replacing an addition with a multiplication.

We had fun once with the second one with some grade 8 students for whom the square root was still a bit of a mysterious object.

In the third one we are canceling the 6's. "Can you *do* that?"

The fourth one is crazy. We all know that  $4^2 = 2^4$ , but what are other examples?

$$4\frac{4}{3} = 4 \cdot \frac{4}{3}$$

$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$$

$$\frac{64}{16} = \frac{4}{1}$$

$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}}$$

[Missteaks Teaching Manual](#)

## Money Machine.

You have access to a magical money machine. You can put in any amount of money you want, between 0 and \$10, and pull the big brass handle, and some “payoff” will come pouring out. Now this payoff depends on the amount you put in. If you feed it  $x$  dollars it will pay out  $y = f(x)$ , and the graph of  $f(x)$  is given at the right.

For example, an input of \$2 will yield a payoff of \$4, whereas an input of \$5 will yield a payoff of \$8. Just to be clear, if you put in \$5 and get \$8 back, your profit or net gain is  $8 - 5 = \$3$ .

Note that pennies are allowed; for example it turns out that an input of \$5.43 yields a payoff of \$8.27. Of course the graph you are given does not have enough fine detail for you to obtain those exact numbers.

The problem is: how should you play to maximize your profit? What is your optimal input  $x$ ?

After thinking about the problem a bit, you might well start to wonder what sort of access you have. Can you play only once? Can you play many times? How many times?

Well, I will give you two options and you can choose.

- *Option A.* You are allowed to pull the handle a total of 100 times.
- *Option B.* The total amount you feed into the machine cannot exceed \$300.

Which should you choose? In each case, what is the optimal value of  $x$  and what is your total profit?

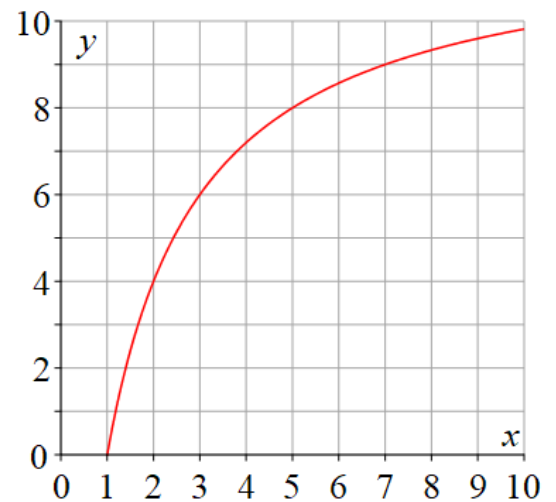
Graphically this will give the student experience in what might be called qualitative thinking in mathematical modeling. Indeed the techniques we will use here are found in applications to the social and biological sciences in which we often don't have equations for our functions, but have only qualitative information about the shape of the graph.

Algebraically we will find a simple equation for the graph and then we will use some of the techniques we have already developed in our work with parabolas.

To begin, we tackle this problem using graphical arguments, and after that, we use algebraic tools to verify our results.

Both approaches are quite instructive.

[Money Machine Teaching Manual](#)



This is an excellent grade 11 problem. It develops strategic thinking and it fosters the connection between algebraic and graphical analysis.



One Arm Bandit-- a drawing by Freder Pastora

## Parabola Tangent

<https://www.desmos.com/calculator/mnlbykbdyl>

I'm going to start with the parabola  $y = x^2$  and then move it so that its vertex is at  $(1, 1)$ .

Then I'm going to draw a line through the origin that is tangent to the new parabola.

The question is, what is the slope of the line and what is the point of intersection? Provide your answers in the simplest possible form.

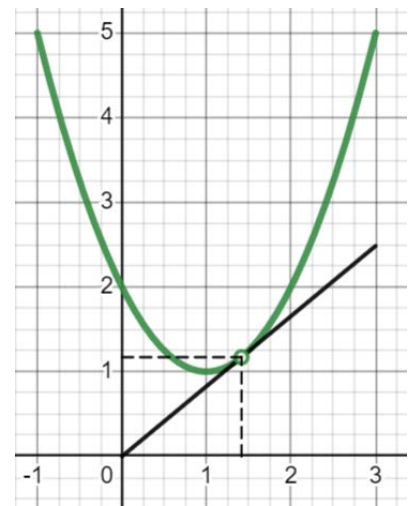
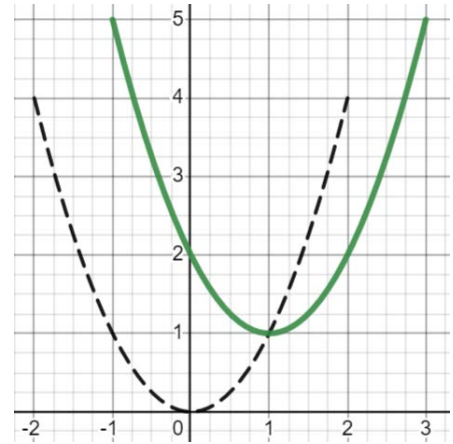
This is a good exercise with the quadratic formula, perfect for Grade 10. The idea is that of all the lines in the family

$$y = mx$$

we want the line that has exactly one intersection with the parabola.

[Of course there are two possible values of  $m$ . What is the interpretation of the other one?]

[Parabola Tangent Teaching Manual](#)



## Parent Game

This game has been extensively studied in the biological and economics literature. It is called *the parent game* as the players can be imagined to be two parents (human or animal) raising an offspring which carries half of each of their genes. The offspring is the benefit and it is of equal (genetic) value to them both. The cost is the complex array of investments made by each parent, often of a different kind, and often with different timing. Payoff to each player is the common benefit minus the cost of the player's investment.

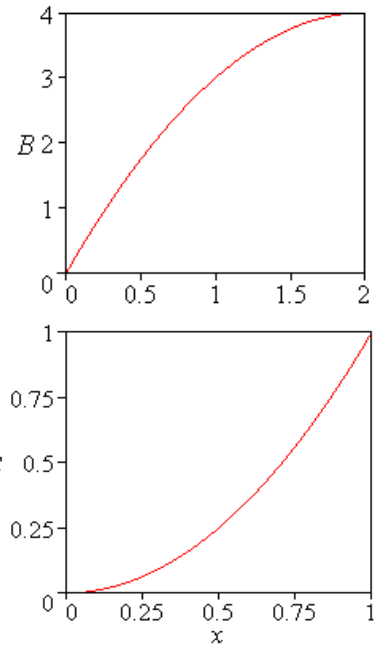
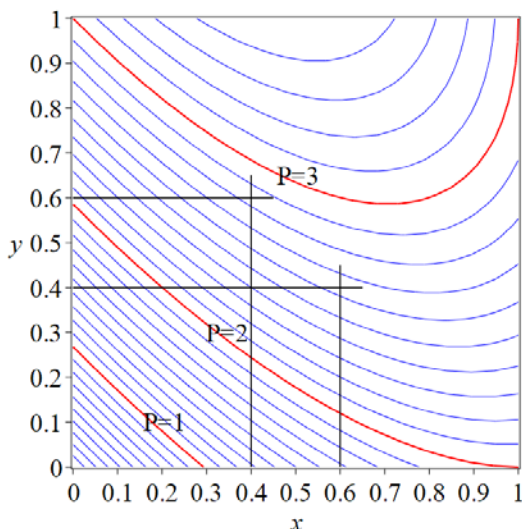
This is a remarkable game in a number of ways. It has good math, good thinking, and an interesting structure. As usual we will analyze this game working with the equations and also working with the graph. But what makes the graphical analysis particularly interesting is that a player's payoff depends on the strategy of both players so it is a function of two variables. We will use a contour diagram to analyze the evolution of the strategies.

We have two players X and Y. They each need to make an investment ( $x$  and  $y$ ) between 0 and 1. Having done that, each player gets a benefit  $B$  that depends on the *sum* of the investments,  $x+y$ , and must pay a cost depending on its own investment.

If we have formulae for the  $B$  and  $C$  curves, we can use these to do an algebraic analysis of the game.

But we can also plot the level curves of the payoff function  $P(x, y)$  which is the payoff to X (investing  $x$ ) with partner Y (investing  $y$ ). This can be used to analyze a number of common variations:

- when the players decide to cooperate
- when the players are chosen randomly from a large population
- when one player has to declare its investment before the other needs to decide.



The queen ant must outwit her diminutive workers to produce enough sons.

Research has shown that a version of the parent game is played in ant colonies. Here the two players are the Queen, who lays all the eggs. And the workers (who are the Queen's daughters) who care for the offspring. The contest here is over the sex ratio of the offspring. The Queen wants 50-50 (males-females) whereas the workers want 25-75. The data suggest that the Queen wins.

## Piece of Wire

Start with the graph

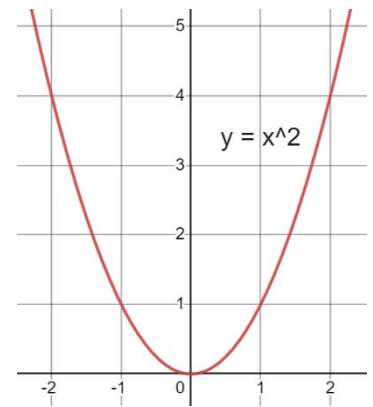
$$y = x^2$$

Your job is to cut out a finite piece of that graph and move it (translation) so that it becomes a curve running between the points

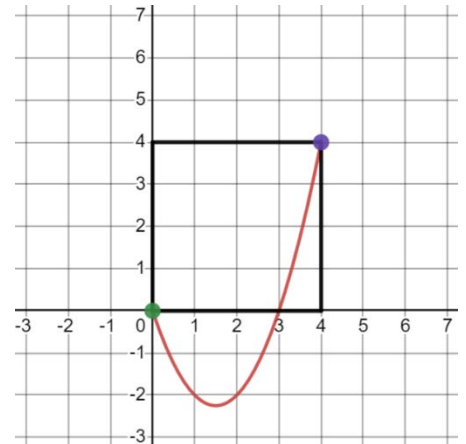
(0, 0) and (4, 4).

Find the equation of that transformed curve.

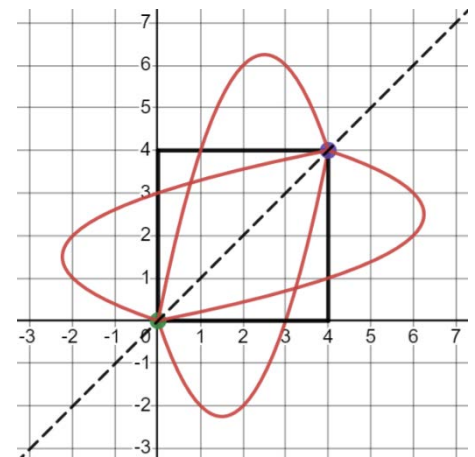
<https://www.desmos.com/calculator/qanemngvs8>



This activity should fit squarely into the grade 11 curriculum. On the one hand it has them moving and reflecting parabolas, but also there's some structural problem-solving to be done—how to find the equations for the diagram.

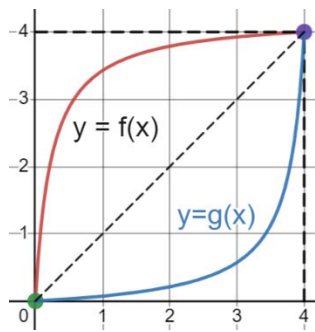
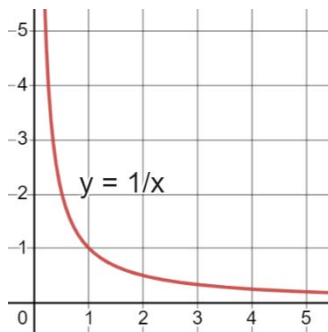


Suppose that the curve is a stiff piece of wire that you can lift up and reposition in different ways so that its end points are at (0,0) and (4, 4). How many different curves can you get? We ask the students to find these and plot them on Desmos.



Following that, we embark upon a similar challenge with the hyperbola:

$$y = 1/x \quad \{x \geq 0\}$$



[Piece of Wire Parabola Playbook](#)

[Piece of Wire Hyperbola Playbook](#)



## Polygon-5

For which  $N$  is there a polygon in 3-space with  $N$  equal sides and all angles  $90^\circ$ ? [Math. Mag. 55, p. 47, 1982]

### Even $N$

Students quickly discover the "staircase" model which provides examples for all even  $N \geq 4$ .

### Odd $N$ .

Clearly  $N = 3$  is not possible. The next case  $N = 5$  is challenging and that's in fact the problem we solve here.

There is a nice construction for  $N = 7$ . First notice that a  $(1, 1, \sqrt{2})$  triangle has a  $90^\circ$  angle at its vertex. We find a way to use such a triangle to form the "top" of our 7-gon.

This example generalizes, by lengthening the staircase, to all odd  $N \geq 7$ . And that leaves  $N = 5$  as the remaining unsolved case.

### $N = 5$ .

We find that the students spend a considerable time trying to build a geometric model, and some will be convinced that they've virtually got it, just a little adjustment here and there. But it's hard to be sure you can get everything right.

One of my former PhD students, Doug, who went on to have a fine career as a high-school teacher and Math Head in Athens, ON, produced a fine Euclidean argument, but my memory is that it was difficult and a bit complex. Euclidean 3-space can be a tough country.

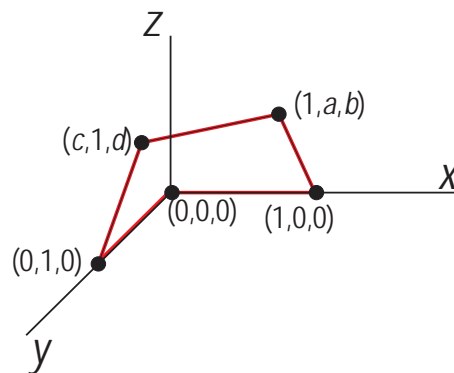
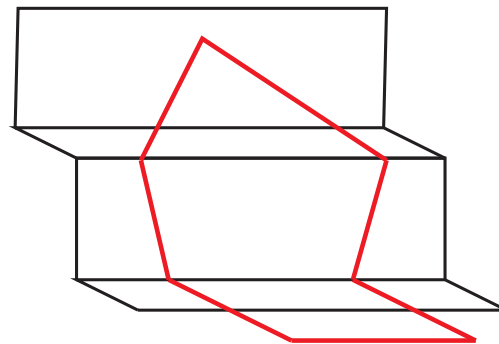
### Analytical Geometry

A much more hopeful approach is to use analytical geometry, that wonderful invention of the philosopher and mathematician Rene Descartes (1596-1650). The idea here is to introduce a system of coordinates and keep track of the vertices of the polygon with their coordinate triples. If we can write down a system of equations which expresses the condition that the angles be  $90^\circ$ , then we have reduced our geometric problem to the unglamorous(?) but reliable(?) and powerful(!) world of algebra.

### [Polygon-5 Teaching Manual](#)

This is a fun, hands-on cross between analytical geometry (grade 10) and vectors (grade 12).

For this session I bring a large supply of plastic straws (cut to equal lengths) and pipe cleaners for the  $90^\circ$  bends.



## Polynomial Play

<https://www.desmos.com/calculator/dton4uw4is>

For different values of the parameter  $a$ , we get different parabolas of the family

$$y = x(x - a).$$

A number of members of this family are drawn at the right. Show that the set of vertices of this family lies along a parabola, and find the equation of this parabola.

This is a simple basic problem, just right for a grade 10 student who is coming to terms with “the equation of a curve.”

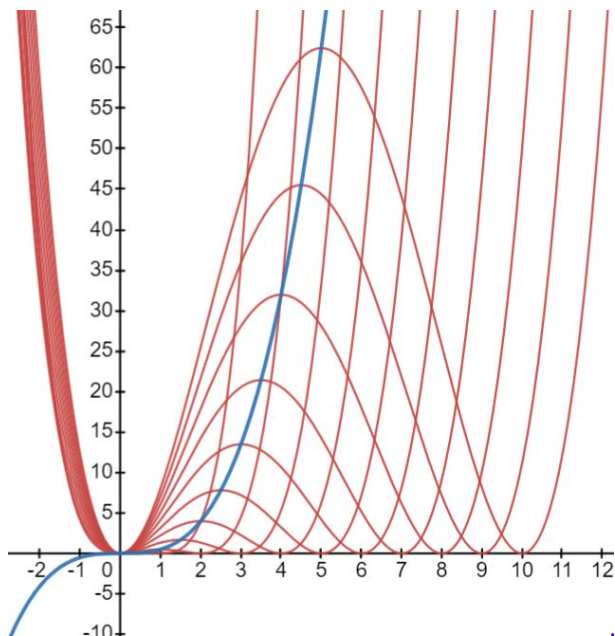
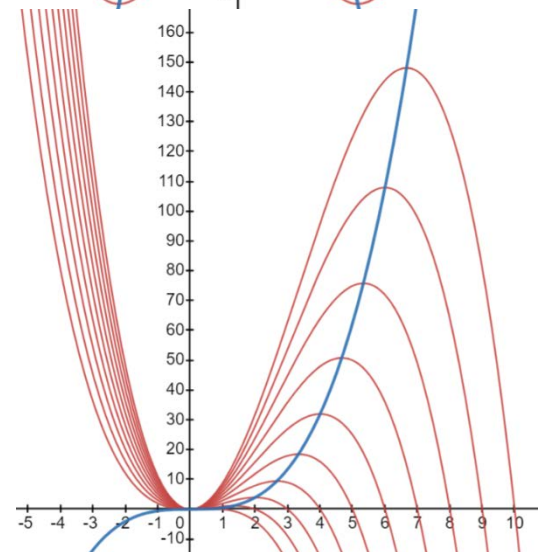
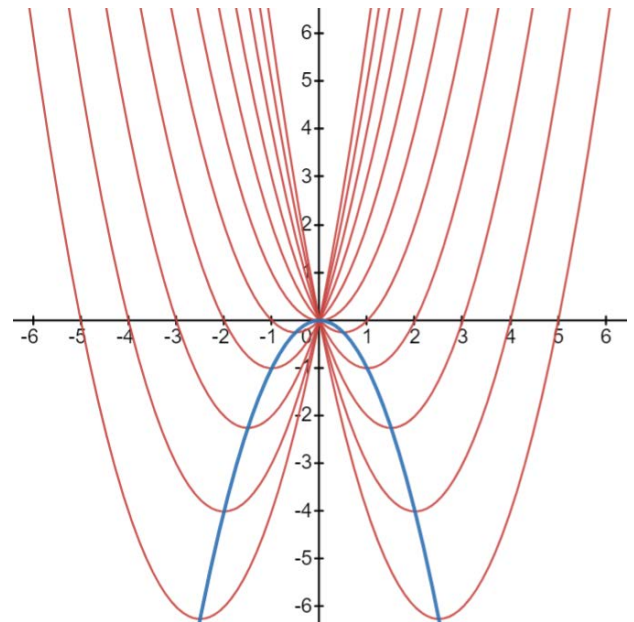
<https://www.desmos.com/calculator/udkivqyv7l>

For different values of the parameter  $a$ , we get different members of the cubic polynomial family

$$y = x^2(a - x).$$

A number of members of this family are drawn at the right. The set of vertices of this family are seen to lie along a curve. Find the equation of this curve.

This is just right for a grade 12 calculus class. However there is also simple polynomial solution that can work well at the grade 11 level.



<https://www.desmos.com/calculator/jtlcns3fvf>

And while we are playing this nice game, let's try a quartic family.

$$y = \frac{x^2(x - a)^2}{a}.$$

Actually, here's a fun problem: create your own family of vertices!

## Polynomial Tangents

At the right, we have a line with  $y$ -intercept  $b = -1$ :

$$L(x) = mx - 1$$

that is tangent to the parabola

$$P(x) = -(x - 3)(x - 7)$$

at a point  $x = a$ . Your job is to find that point of tangency.

There are many ways to do that. Certainly calculus is one of these, but another is to take the family of all lines with  $y$ -intercept  $-1$  and choose the line that intersects the parabola exactly once.

<https://www.desmos.com/calculator/3jo9gctzx2>

But here we will look at a method that is simpler than both of these, and we will see that it can be used to solve an even more complex type of problem.

What we ask the students to do is to try to sketch the graph of

$$L(x) - P(x)$$

which is the vertical distance between the line and the parabola.

Well, the graph appears at the right. Qualitatively it's what most of the students come up with, but one aspect they almost all miss is the bilateral symmetry. You don't expect that looking at the original. Of course, since the resulting expression has to be a parabola with vertex at  $x = a$ , so it has to be symmetric. Indeed its equation, simplified, is

$$\begin{aligned} D(x) &= L(x) - P(x) \\ &= mx - 1 + (x - 3)(x - 7) \\ &= x^2 - (10 - m)x + 20 \end{aligned}$$

Okay, right away I know that  $a^2 = 20$ . Why is that? Well it's because I know that the  $D$  equation also has the form:

$$D(x) = (x - a)^2 = x^2 - 2ax + a^2$$

*Viola!*

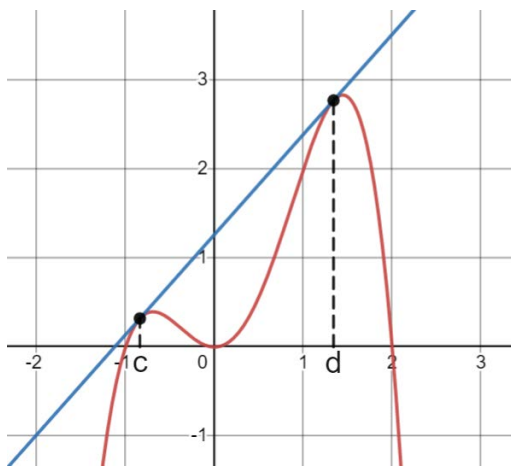
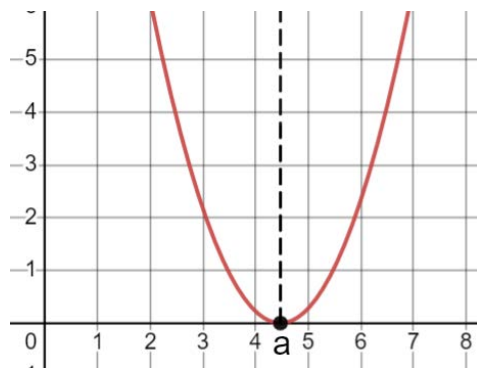
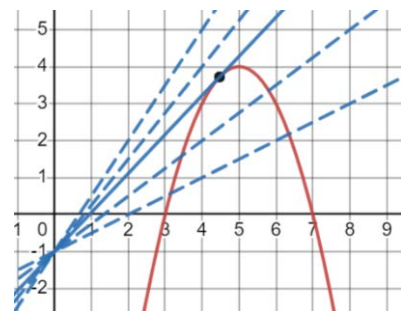
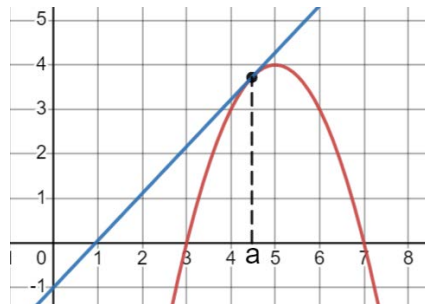
Okay: here's an amazing problem. At the right is drawn the graph of the quartic polynomial

$$Q(x) = -x^2(x + 1)(x - 2)$$

and a line tangent to the graph at two points  $c$  and  $d$ .

<https://www.desmos.com/calculator/sxj6prpahc>

Find the equation of the line. Wow.



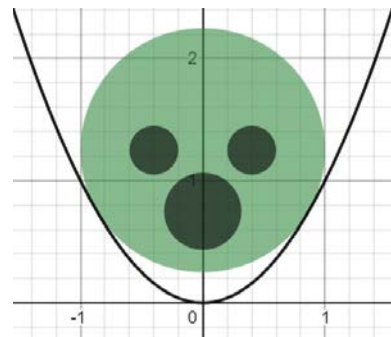
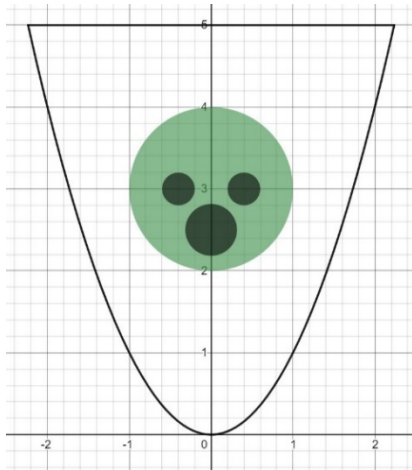
[Polynomial Tangents Teaching Manual](#)

## Quin Bouncing

<https://www.desmos.com/calculator/tb7grsodmb>

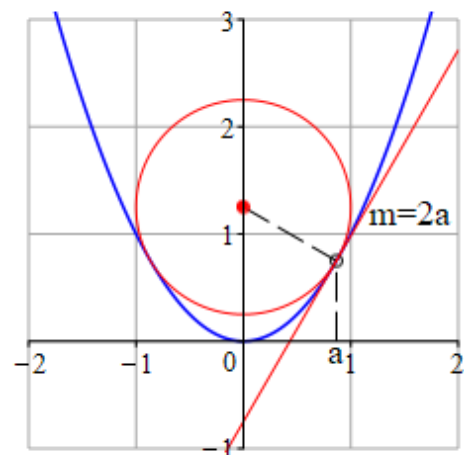
Quin travels up and down the  $y$ -axis, bouncing between the line  $y = 5$  and the parabola  $y = x^2$ .

Your task is to build the animation and a key part of that is to identify the exact point at which Quin hits the parabola and bounces back up.

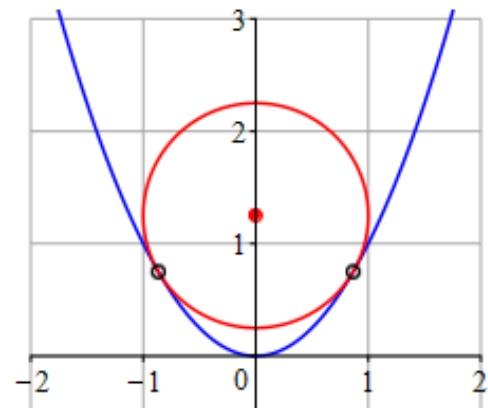


This is an interesting problem and there are number of ways to solve it. We will look at two of these.

1. *Geometry.* The first comes from calculus but we sidestep the calculus by simply telling the students that the slope of the parabola  $y = x^2$  at any  $x$  is  $2x$ . This leads to a lovely geometry solution, perfect for grade 10. But it also works well in grade 11.



2. *Algebra.* The second studies the descending circle and keeps track of the number of intersections between the circle and the parabola. It goes from 0, to 2, to 4, and we want that transitional point where there are exactly 2. This solution belongs more at the grade 11 level.



[Quin Bouncing Teaching Manual](#)

## Redball Blueball

At time  $t = 0$  a red ball is projected from the origin and travels along the line  $y = mx$  at constant speed 2. At the same moment a blue ball is dropped from the point  $(4, 4)$  and travels down the line  $x = 4$  at constant speed 1.

The problem is to find the value of  $m$  (the slope of the red line) that will produce a direct collision of the two balls.

The analysis can vary depending on what is taken as the unknown. Two possibilities are the slope  $m$  and the  $y$  coordinate of the intersection point. The latter seems a bit simpler.

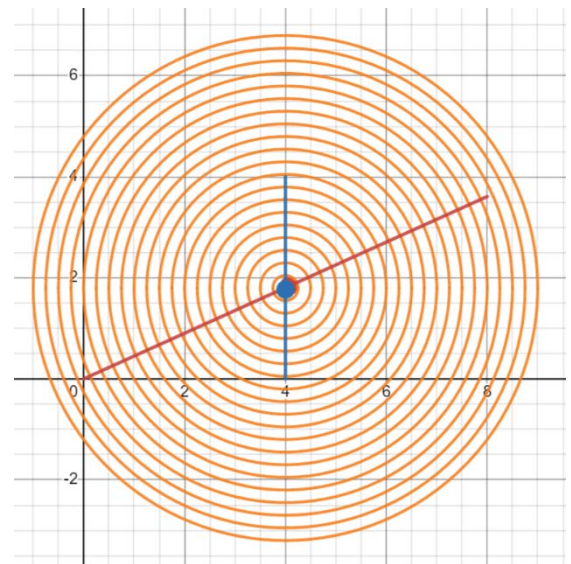
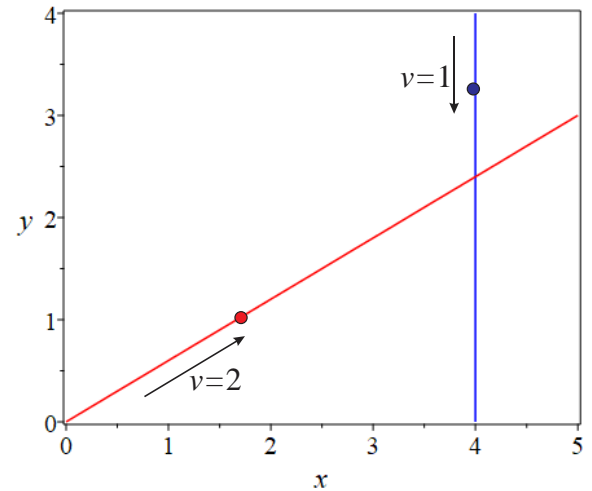
For students with some Desmos experience, we ask them to construct the animation, using a slider to set the slope  $m$  of the line. It can take some careful thought to get the red point to move at speed 2.

<https://www.desmos.com/calculator/o7ewmtknfu>

A nice free-for-all activity is to invite the students to design an “explosion” to occur when there is collision.

<https://www.desmos.com/calculator/hwatgplq8o>

[Redball Blueball Teaching Manual](#)





## Roofing

Here is a sad house with no roof. In fact the roof is lying at the side waiting to be installed. Your job is to move it into place.

Of course you will first need to find its equation. What you know is that the graph is a member of the family of curves:

$$xy = k^2.$$

Looking through the drawings of the house, you find a schematic of the installed roof showing the vertical distance between the peak of the roof and the top of the house to be  $\sqrt{2}$ .

That, together with the fact that the house has width 6, should give you all the information you need to find  $k$ .

We have found that there is more than one way to do the analysis—some algebraic and others more graphical. We will look at an elegant geometric analysis that a couple of high school students came up with.

Once you have found  $k$ , you need to install the roof. To do that, you will make use of your faithful Desmos crane.

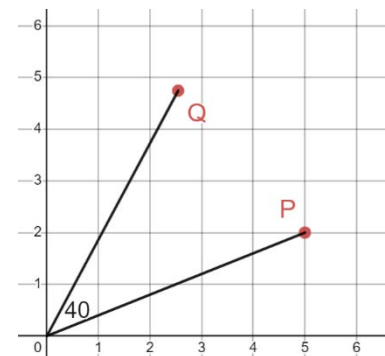
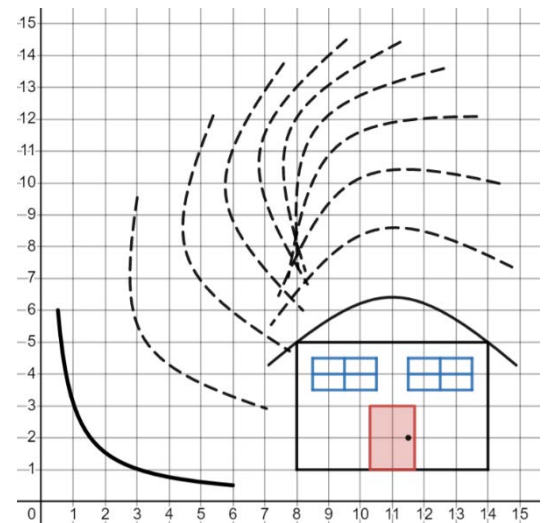
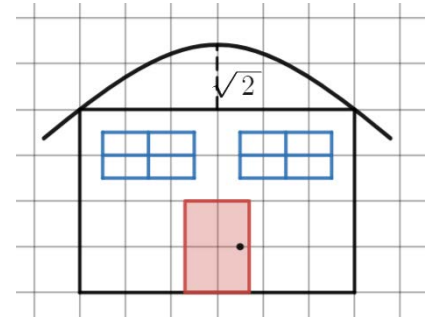
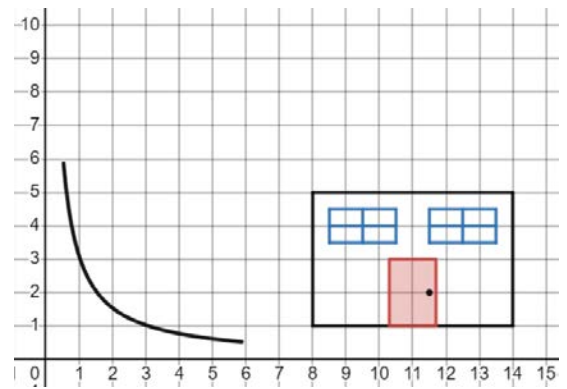
Now this is a bit of a complex system. You need to move the roof, and at the same time, gradually rotate it clockwise through  $135^\circ$ .

We found that to arrive at a simple transparent program, we needed all our skills of organization and design. Not to mention clear thinking.

One thing we found useful was to extend our rotational repertoire. What we don't yet have is a direct way to rotate a given point through a given angle about the origin. For example, take  $P(5, 2)$  and rotate it through  $40^\circ$  about the origin. What are the coordinates of the rotated point?

That seems to be a simple enough question, and we could probably figure it out if we had to, but maybe there's a simple formula.

Indeed there is, but (believe it or not) it's generally not done till university. We fix that, and then we use the formula to rotate the roof.





## Rotating Table

A circular table has 4 pads, equally spaced around the perimeter, and each covered with an opaque screen. Beneath the screen, the pads are either red or blue. Every minute the table rotates for a few seconds at high speed.

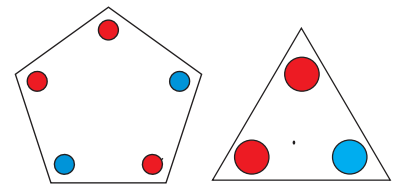
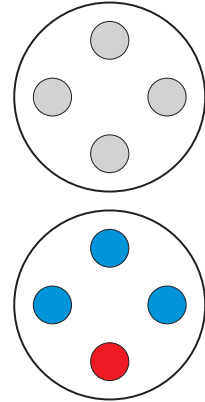
During each interval between rotations, you are allowed to touch two pads, your choice. The screen over those two pads will open allowing you to see the colours of the pads. At that point you can, as you wish, change either or both of the two colours.

If at any point the pads all have the same colour, a bell rings and you have "won." The problem is, can you find a strategy which will guarantee a win in a declared finite number of moves?

Needless to say, the table rotates so quickly that you cannot follow the pads that you last opened.

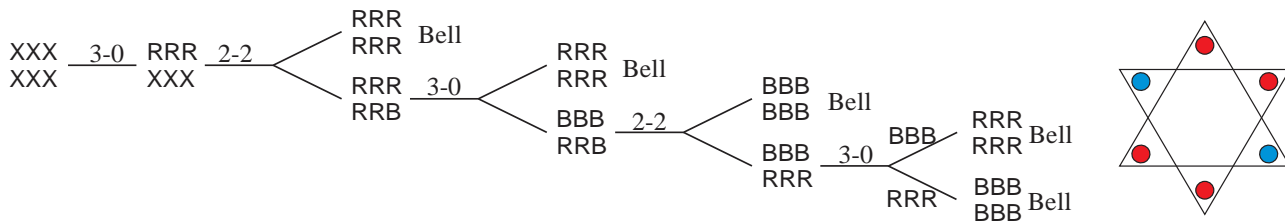
One great thing about this problem it is that there are easier variants that you can cut your teeth on. We can vary

- $n$  the total number of pads
- $m$  the number of pads you can open.



open $m$	total# pads $n$				
	2	3	4	5	6
1		L			
2		W			
3					
4					
5					

Working in small groups, the students can see how many of the slots they can fill with either W or L. To put an L they need a configuration that cannot guarantee a win. To put a W they need an algorithm that will eventually win. For example the algorithm below will produce a win for  $n = 6, m = 4$ . Last time we did this with a group of high school students, this was displayed on the board by a grade 11 student.



Interestingly enough, there is an extraordinary theorem (Lewis and Willard 1980) that provides exactly the  $(n, m)$  pairs that will win. A winning strategy exists precisely when

$$m \geq \frac{p-1}{p} n$$

where  $p$  is the largest prime divisor of  $n$ . For example, the largest prime divisor of  $n=6$  is  $p=3$ , and the condition says that a winning procedure will exist when  $m$  is at least  $\frac{2}{3}6 = 4$ .

Go figure.

## Skunk

This is a simplified version of the game "skunk" that is popular in elementary school math classrooms. It's a good game to play in class and let the students talk about their strategies.

Here's how the game is played. To start with you stand. Two dice are thrown. If neither of them shows a 1, you get the sum of the dice as your score. But if a 1 comes up the game is over and your payoff is zero. If the game can continue you decide whether to leave the game (sit down). If you do, you get your current score as a payoff. If you stay in the game, the process repeats. The dice are thrown again and if neither of them shows a 1, you *add* the sum of the dice to your current score and again you decide whether or not to stay in the game. And everything continues as before.

The problem is to find the strategy which maximizes your average payoff per game.

The students discuss the different strategies they have been using. Two factors emerge in the decision of when to leave the game: the current score and the number of rolls so far. There is quite a lively debate over which of these are relevant. In fact, only the current score is relevant but some students are hard to convince. Some will argue that if a 1 has not come up after many rolls, it is more likely to come up on the next roll.

Of course we will need to know the probability that a 1 will come up. The students will have a good discussion about that but at some point someone will suggest looking at the 36 possibilities. It's important to note that these are all equally likely.

The analysis of the game leads to quite an elegant argument. If I currently have score  $s$  then I should leave the game if my expected score  $s'$  next time is less than  $s$ .

The tree diagram at the right calculates  $s'$  as an average of the two possibilities: either a 1 comes up, or it does not.

$$s' < s$$

$$\left(\frac{11}{36}\right)(0) + \left(\frac{25}{36}\right)(s + 8) < s$$

This simplifies to give:

$$s > \frac{200}{11} = 18.2$$

As soon as your score exceeds 18, you should leave the game.

[Skunk Teaching Manual](#)

For example, for the sequence of rolls:

(2, 5)

(4, 2)

(6, 1)

Those who sit after the first roll get payoff 7.

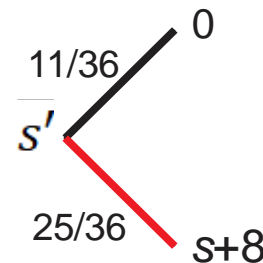
Those who sit after the second roll get payoff 13.

Those who stay standing for the third roll get payoff 0.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 36 equally likely possibilities--11 of these have a one. So the probability of getting a one is 11/36.

The table also provides a nice way of seeing that the average outcome, if neither die has a 1, is 8.



## Spinners

I have two spinners with 3 colours, Red, Blue and Gold:

Spinner S1	
R	1/3
B	1/3
G	1/3

Spinner S2	
R	0.45
B	0.35
G	0.2

In S1, the colours are equally weighted,  
<https://www.desmos.com/calculator/okvlmpwymf>

In S2 the colours are differently weighted.  
<https://www.desmos.com/calculator/sniekpfzfxo>

Here's the game. I choose one of the spinners and spin 10 times. I get the data sequence:

$$D = [R B B G B B R G R B].$$

Which spinner did I choose?

This problem takes off on a fascinating journey. What we find students do at first is calculate the probability of getting that sequence of spins using S1 and then using S2, and they compare those.

For example using S2, the probability of getting the red on the first spin is 0.45, and the probability of getting blue on the second spin is 0.35, etc. These can be multiplied together as the spins are independent events.

Now what is this calculating? We look carefully and we remind the students about the WonderCards. And they discover that what they have calculated are

$$P(D | S1) \text{ and } P(D | S2).$$

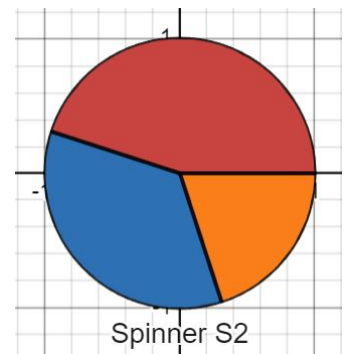
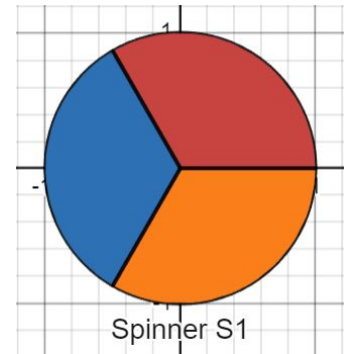
*But what did I ask for?!* Well that takes a bit of thought, but they soon realize that I wanted:

$$P(S1 | D) \text{ and } P(S2 | D)$$

Those are the *reverse* conditional probabilities! Not what I asked for.

Time for Bayes Formula.

[Spinners Teaching Manual](#)



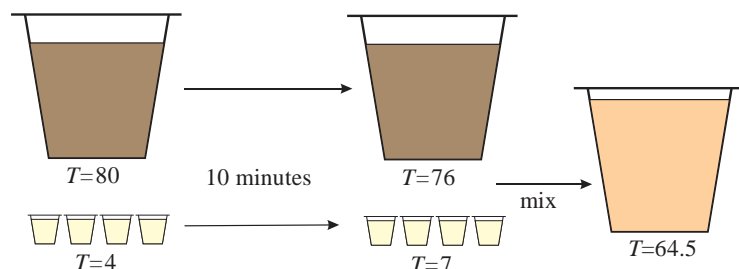
## Timmy's (Coffee&Milk)

You buy a large coffee (300 ml) at 80° and four milks (15 ml each) at 4°, and get into your car which stays at a constant temperature of 16°. It will be a 10-minute drive before you can drink the coffee, and you want it to be as hot as possible at that time. Do you put the milk in at the beginning or after the 10 minutes when you're ready to drink it? Do you mix first and then drive or drive first and then mix? As an example, let's look at the latter.

*Mix Last.* What happens to the temperature of each if they are kept separate for the 10-minute drive? Well, the coffee drops from 80° to 76° and the milk rises from 4° to 7°. What then happens when you mix them?

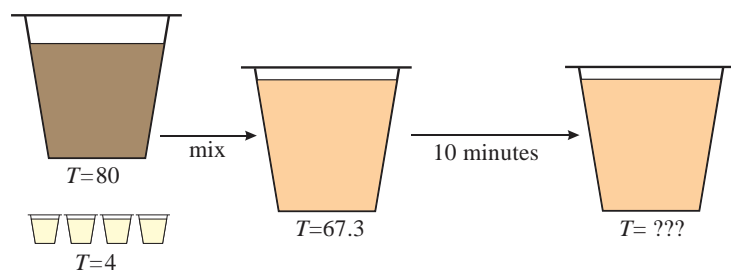
Then when we mix them, we ought to get some sort of average of 76 and 7. But what sort of average? Well, as you might suspect, it should be a weighted average where the weights are the different amounts of coffee and milk:

$$T = \frac{300(76) + 60(7)}{360} = \frac{23220}{360} = 64.5$$



*Mix First.* To mix them at the beginning, we use the same weighted average approach but with the starting temperatures:

$$T = \frac{300(80) + 60(4)}{360} = \frac{24240}{360} = 67.3$$



But then what? The only temperature change information we have so far pertains to the coffee and milk separately. Is that of any help to us here?

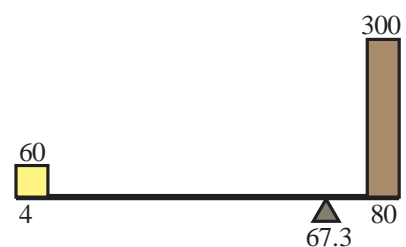
Indeed it is. But to make effective use of them, we need Newton's Law.

[Timmy's Teaching Manual](#)

This is a good grade 11 or 12 problem. It requires facility with exponential decay and the use of the exponential multiplier. The leaking tire model provides a good preparation for this.

But it also embeds these mathematical ideas in a complex story that requires the students to keep track of the different pieces and the roles that they play.

And of course it features Newton's Law of Cooling.

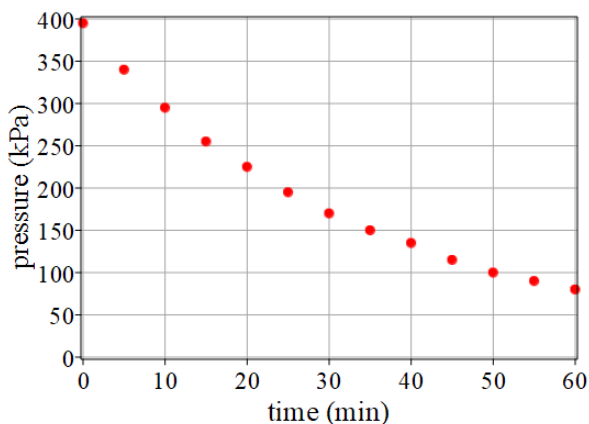
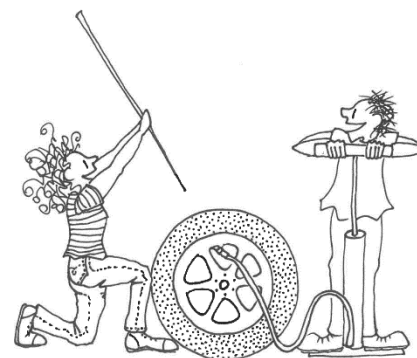


I've worked many problems with teachers over the years. And when I meet them at a conference or a street corner, this is often the problem they remember. Why?

Because they've "been there, done that."

## Tire

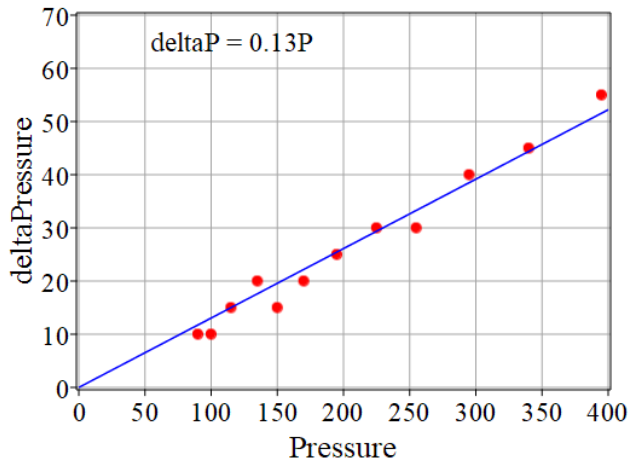
If you can manage to find an old car-tire on rims, bring it to class, fill it to 400 kPa, and let the kids punch a hole in it. Below are the data we collected in 1998 at KCVI.



Is this a process of exponential decay? There are lots of ways to investigate this. One of these is to calculate the pressure *change* over each 5-minute period and see if this is proportional to  $P$ .

One way to do that is to plot  $\Delta P$  against  $P$ , and see if they are proportional. That's the graph below and there's a strong linearity.

time $t$ (min)	pressure $P$ (kPa)	change $\Delta P$
0	395	55
5	340	45
10	295	40
15	255	30
20	225	30
25	195	25
30	170	20
35	150	15
40	135	20
45	115	15
50	100	10
55	90	10
60	80	



This is a good grade 11 analysis. When we get to grade 12 we will do this analysis in a more powerful way--plotting the logarithm of  $P$  against time and again expecting a straight line.

The best fit line (through the origin) has slope 0.13, giving us:

$$\Delta P = 0.13P$$

Every 5 minutes,  $P$  is decreased by 13%. Thus every 5 minutes  $P$  is multiplied by  $m = (1 - 0.13) = 0.87$ , so that the one-minute multiplier for  $P$  is:

$$k = m^{1/5} = (0.87)^{1/5} = 0.973$$

and our model for the pressure at time  $t$  is

$$P(t) = P(0)k^t = 395(0.973)^t.$$

This is our exponential decay formula for tire pressure. It can be plotted on top of the experimental data and it is seen to give a good fit.

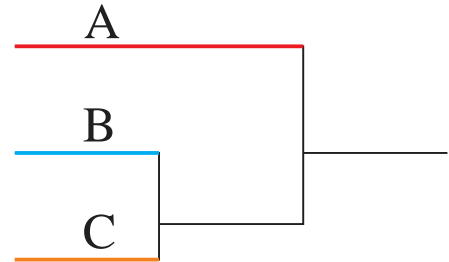
[Tire Teaching Manual](#)

**Tournament**

A, B and C are playing in the simple tournament diagrammed at the right. B and C play first and then the winner plays A.

Data from past matches give us the following predictions:

- If A plays B: A will win 70% of the time
- If A plays C: A will win 80% of the time
- If B plays C: B will win 60% of the time



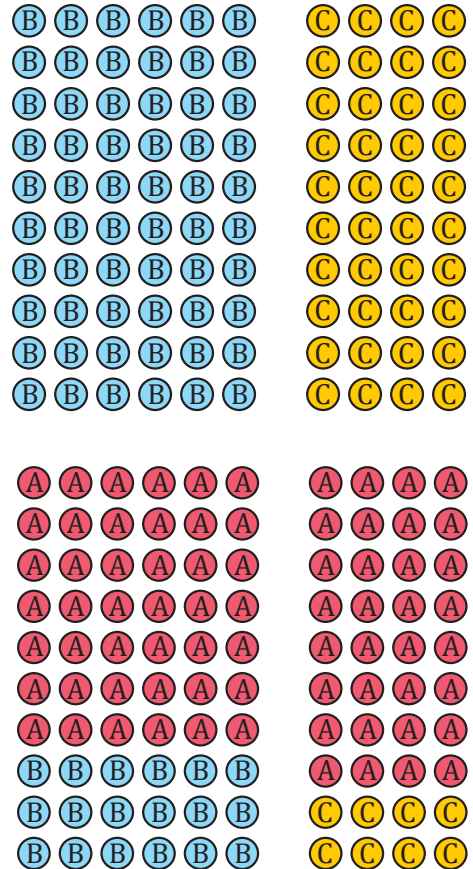
Based on these data, calculate the probability A will win the tournament. Use an effective diagram to illustrate your argument.

This is an interesting problem in that there are number of ways to think about it, and do the analysis.

In a way this problem provides a gentle introduction into the world of conditional probability, a topic that we in RabbitMath feel is exactly right for grade 9.

Here is a nice visualization from Dave. If the tournament is played 100 times:

- the first diagram depicts the wins after the B-C round.
- the second diagram depicts the wins in the last round.



*Colin's "instant" solution*

If B and C were equal, A would play either of them half the time so A's probably of winning would be halfway between 70 and 80, so 75%.

But B and C are not equal: they are 60:40 instead of 50:50 so divide 70 and 80 in the ratio 6:4 so it's 74%.

*Why not 76?*

Because A has more chance of playing B the harder player, so it goes down from 75.



## Trains

Parallel structures--this presents the students with two worlds:

*The trains world.* We are building trains of a fixed length using cars that are either length 1 or length 2. A sequence of integers with a concrete realization (number of trains of a fixed length) giving us a powerful access to the surprising patterns of the Fibonacci world. For example, here is a train of length 12.



It uses six 1-cars and three 2-cars but different numbers of cars in different orders are possible. How many different kinds of trains are there?

*The Fibonacci world.* A sequence of integers generated with one of the simplest of all recursive generators:

$$x_{n+1} = x_n + x_{n-1}$$

This is an abstract world which contains many unexpected patterns, some of which we will encounter here.

*Problem 1.* I want to construct a train of total length  $n$  units using cars which are either 1 unit long or 2 units long. The question is, for each value of  $n$ , how many different trains are there? [Answer: 233].

The answer to this problem turns out to be the Fibonacci numbers. One can establish this using combinatorics but there is an elegant recursive argument that is perfect for grade 10.

*Problem 2.* Take two consecutive Fibonacci numbers and add their squares. We always get a Fibonacci number. For example:

$$t_4^2 + t_5^2 = t_{10}$$

This is not easy to prove. Go ahead and try!

But there is a wonderfully simple proof using trains.



1

2

Length of train	Number of trains
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34

$n$	$t_n$	$t_n^2$
0	1	1
1	1	1
2	2	4
3	3	9
4	5	25
5	8	64
6	13	169
7	21	
8	34	
9	55	
10	89	

[Trains Teaching Manual](#)

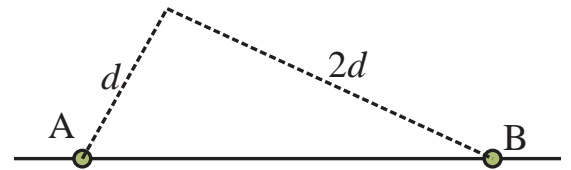
## Two Trees



I am wandering on the grasslands and I notice two tall trees in the distance that appear to me to be exactly the same height. Now I know those trees. I call them A and B, and it is known that B is exactly twice as tall as A. To amuse myself I decide to try to walk towards the trees in such a way that they always appear to me to be the same height. What can you say about the shape of the path I will have to follow?

Well since B is twice the height of A, every point on my path will need to be twice as far from B as from A.

That's pretty much all I give the students, and right away they begin to make sketches.



[Two Trees Teaching Manual](#)

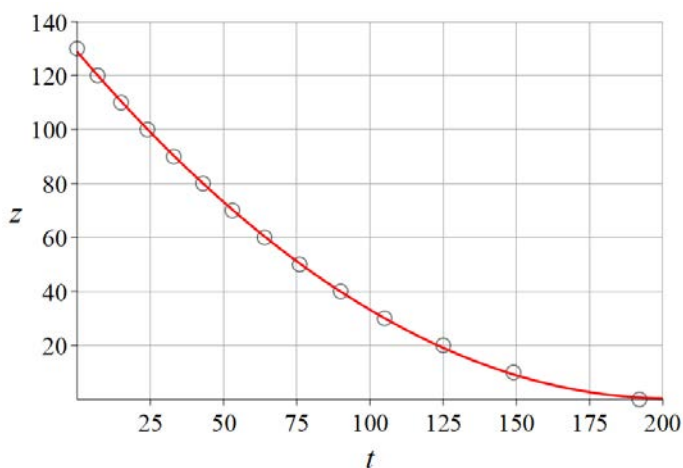
## Water Tank

A cylindrical bottle contains 900 ml of water. At  $t=0$  (minutes) a hole is punched in the bottom, and water begins to flow out. It takes exactly 100 seconds for the tank to empty. Draw the graph of the water level  $z$  in the tank against time  $t$ . Explain the shape of the graph.

The students rightly predict the concave-up shape of the graph and even conjecture that it's a parabola. It turns out that they are correct, and the reason behind that (Newtonian mechanics) is a good story for grade 12. But the activity discussed here is great for grade 10 and even 9. The formula we get for this graph is

$$z = 0.09(t - 100)^2$$

Following this, we bring a large soft-drink bottle to class and collect and plot the data. We get what looks like a pretty good parabola and Excel or Desmos give it a good fit.



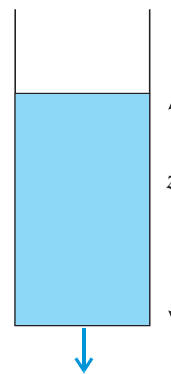
But there's something mysterious lurking behind the goodness of this fit. If we take the square root of the formula above we find that  $\sqrt{z}$  is a *linear* function of  $t$ , and should deliver a straight line. Well it does, but that line tells us a revealing story.

The first graph at the right plots the square root of the data and the best-fit line and it looks pretty good except at the end when the tank is nearly empty. The very last data point is definitely below the line and even seems to be pulling the line down at the end. That is, it doesn't "fit" the trend established by the previous points.

The second graph removes that "outlier," gives a better fit to the rest of the data, and predicts the point where the tank should become empty.

So what went wrong at the end? It's called viscosity or surface tension.

[Water Tank Teaching Manual](#)



time $t$ (s)	height $z$ (mm)
0	130
7	120
15	110
24	100
33	90
43	80
53	70
64	60
76	50
90	40
105	30
125	20
149	10
192	0

