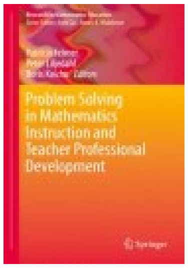

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Chapter 5

Reforming school mathematics: two levels of structure

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Abstract. Many articles and papers over the past 100 years have suggested that mathematics education has lost its way in a number of critical respects. One indication of this is certainly the hugely destructive debate between discovery and drill, a consequence of which is an emphasis, throughout the school curriculum, on technical routines.

For me, mathematics is the abstract study of structure. The structures that mathematicians choose to work with have sophistication and beauty and it is remarkable that these same structures arise in art, in nature, and in the physical and even social sciences. So often, it is by following the beauty that we are led to the truth, and mathematics is a powerful showcase for this delightful principle. But in spite of a century-long call that school math attend to this vital aspect of mathematics, an emphasis on structure and beauty, for example in the choice of material, is notably absent from realized curricula.

My view is that such a curriculum change cannot happen without a change in the very structure of the curriculum. Quite simply, we must put aside our technical wish-list, and select for our students activities and problems that give them a true mathematical experience, and then build the curriculum from there. Thus this article is about structure at two different levels, the first is the structural richness of the mathematical activities I want to see in the classroom, and the second is a new structure for the curriculum itself.

5.1 Some brief historical comments

It has been 100 years since the end of the first Great War. The history of secondary school math education reform during that century has been a tangled tale and I will begin with a summary of some of the main episodes. In this, I will be following two articles, one by Jeremy Kilpatrick (1997) and another by Kate Raymond (2018). The tangled nature of the tale comes from the fact that there were always two forces at work, one narrow and the other wide, but at different times and in different movements, these forces locked horns along different axes. Along one such axis, the narrow view focused on the preparation of students for college and university and ultimately for their participation in technology and the STEM disciplines, while the wider view emphasized the more general humanistic development of informed citizens for a full rich life. Along another axis, the narrow view tended to focus on procedural fluency (back to basics), and the wider on creativity, discovery and conceptual understanding. As a general rule, as we will see, the wider view tended to have less effect on classroom practice than the narrow view. That's not surprising—narrow more focused objectives tend to be easier to grasp and implement.

In general, both views make good sense to me and one would think they could happily coexist. Indeed the oscillations that appear in the historical record often seems to me to be over-reactions to positions that were not as far apart as many seem to have thought. Indeed my, perhaps idealistic, objective in this article is to outline a curriculum structure, one that was long ago elegantly articulated in the philosophical record, that would support both of these viewpoints and be true to the nature of the subject.

Following the first war there was definitely a flowering of a wide view of "math education for all." Philosophically this can be seen in the writings of both Whitehead and Dewey (and more on this later) but as both Kilpatrick (1997) and Raymond (2018) observe, it was also explicit in the 1923 report of the MAA National Committee on Mathematical Requirements. The report argued that

> …the practical aims of school mathematics should be secondary to the mental training and development of skills necessary to the discipline of mathematics and the development of an appreciation for the beauty, power, and logic in mathematics and geometric objects. By focusing on these aims, scholars hoped to avoid school mathematics becoming "a collection of isolated and unrelated details" and instead make mathematics more appealing to a broader range of students. (cited by Raymond 2018 p. 3.)

Raymond goes on to suggest that these ideas appear to have had little effect on classroom practice. The technological growth emerging from the second great war, along with the 1957 "sputnik" wake-up call, promoted along one axis a narrowing emphasis on student preparedness for future scientific and engineering challenges, and along another axis, a widening view of the nature of mathematics, away from procedural fluency towards conceptual understanding (Raymond 2018 p. 4). A dominant idea was that to succeed, students would need a "proper" treatment of mathematics, often interpreted to mean pure math and abstract structures, and this became known as the “new math.”

Of course there was swift reaction and Morris Kline’s 1973 book *Why Jonny can’t Add: the Failure of the New Math* became in many ways the face of the reaction. Kilpatrick (1997 p. 956) notes that “Kline ended the book by arguing that the appropriate direction for any reform ‘should be diametrically opposite to that taken by the new mathematics’ (1973, p. 144), toward mathematics as an integral part of a liberal education, with connections to culture, history, science, and other subjects.*”* But that component of Kline’s message did not catch on and the “back to basics” reaction to the new math won the day. Both Kilpatrick (1997 p 956-7) and Raymond (2018 page 5) argue that the new math movement was far more diverse than is commonly realized and was never properly tested.

In the 1980s the reform movement returned but this time under the formidable banner of the National Council of Teachers of Mathematics (NCTM) *Standards* (1989) which advocated ‘mathematics for all’—the intention of which was to empower all students with the skills and abilities that would enable them to be active, engaged, and critical members of democratic society. After decades of narrowing the focus of school mathematics to prepare students for technological careers, these documents were the first to push back against the limited view of school mathematics and insist on a broader conceptualization. (Raymond 2018, p. 6).

Of course, there was again strong reaction, strong enough that the term “math wars” was used. The main target of the reaction was the “discovery” approach to learning which, at the elementary level, diverted students from the important task of learning multiplication tables and adding fractions and at the secondary level, with its use of heuristics and diagrams, prepared students badly for a rigorous course in university calculus. Indeed the debate had an echo at the university level in the reform calculus movement, which in itself has had a huge effect on first-year university calculus courses today. In the early 1980’s there was a suggestion that the coming world of computer technology might be better served by a course in discrete math or linear algebra rather than calculus and, led by Andy Gleason and others, there was a response to make calculus more relevant and mainstream. That movement was successful in that calculus remains today the default (and often required) first-year university math course. Interestingly enough, in a somewhat altered form, the idea, that calculus might not be the best default course, is now coming back, though in altered form, one that features areas of math and stats that are closer to data analysis.

A central figure in the traditionalist camp was H. Wu of Stanford University. To get a sense of the state of the argument at the close of the 20th century, it is interesting to look at a pair of papers of Kilpatrick (1997) and Wu (1997) which appeared side by side in the *American Math Monthly*, and in fact the last part of Kilpatrick’s remarks focused on the Wu paper. Wu makes a number of interesting points—interesting in that they are well worth discussing. He does accept the appropriateness of reform calculus for the typical science and engineering student, but fears that it will not well serve the student who is destined for serious university mathematics.

Such students "need rigorous mathematical training, and would not be satisfied with a steady diet of persuasive heuristics, graphic displays, and nothing else" (Wu 1997 p 947). I go most of the way with this but would phrase it differently. Students who are destined to study serious mathematics need to be able to make rigorous arguments but I believe that the opportunity to understand and practice these can be given to them in a course that features persuasive heuristics and graphic displays.

5.2 My own half century

For the past 50 years I have been constructing "discovery" problems for high-school students. But over that period there have been a few ways in which my work has changed. At the beginning, I regarded these problems as "after school" enrichment for motivated students. That possibility still exists but, for me, the main stage is now the regular classroom. That objective requires tasks that provide a low mathematical floor (requiring minimal prerequisite knowledge), and a high mathematical ceiling (offering opportunities to explore more complex concepts and relationships and more varied representations) (Gadanidis *et. al.* 2016 p. 236 Boaler 2016 p 115). As I pursue that objective, I find to my surprise that many high-ceiling problems, such as those found in university mathematics, can be engineered to have an invitingly low floor, and can work beautifully in high school.

Over the past few years I have made a deliberate effort to tie my problems to the mandated curriculum, and this has affected my choice of subject matter. For example, for the first few decades I chose problems that were fun, enticing and mysterious, and worked with areas such as geometry, probability, combinatorics, logic, games, puzzles. But in Ontario, fully half of the entire high school math curriculum works with properties of functions, and while I believe that this is unbalanced, my basket of activities has moved somewhat in the direction of functions. But here's an interesting anecdote. In my third-year undergraduate course for future math teachers, I take my problems/activities from a balanced set of areas including the analysis of functions. Towards the end of the course I have group projects and students can choose the problems they want to work with. In 20 years with that course, no student has ever chosen to work with functions. What that tells me is that their own school experience with functions has hardly ever engaged them in play, in design and construction, or in mathematical thinking.

I have always had an eye on the preparation of our secondary school students for university but only recently has that become my main focus. I watch carefully to see what my first-year university students struggle with. That can be hard to perceive, but my feeling is that their struggles seem to be more connected with the focus and clarity of their thinking rather than the execution of what are called "the basics." A related aspect of these struggles is their handling of problems with a complex structure. Complexity can be contrived, and I find that to be often the case in problems that the students are given, but there are also complexities that are organic to the structure of the problem. These are more important, in part because they arise naturally and are thereby closely related to structural complexities that the students will encounter in their own future lives, both professional and personal. In university, students frequently encounter structures with this level of sophistication. but I find almost no problems of this kind in high school mathematics.

What do I do with my ever-growing collection of problems? I show them to the teachers that I know or might meet and ask if they want to use them, or if they would invite me into their classroom to try them out, or better still, let me come and watch while *they* work with them. I do get offers, but the teachers that I talk with are often wary. There could be many reasons for that, but the one typically stated is that they are running short of time. They have after all a curriculum to cover and it can easily require the full 110 hours that the Ministry allocates. Of course my "wonderful" problems are designed to *be* the curriculum, such that nothing else is needed. If the students can do those, they will surely be ready for my first-year calculus and linear algebra courses. But I can't yet promise that because the problems are a long way from being organized into a complete, coherent, well-supported package. So I certainly understand the teachers' hesitation and am grateful to those wonderful colleagues who have been happy to work with me.

But this brings up the question of the nature and the structure of the curriculum. Certainly the curriculum of problems has quite a different structure from the one we currently find in school mathematics. Is it apt to work? Is there anything to be said for such a curriculum? In fact the ideas of some of the greatest thinkers of the past 100 years interact rather well with this question of curriculum structure. I have three of these in mind: Alfred North Whitehead, John Dewey and Seymour Papert (Fig 1).

Alfred North Whitehead
1861 - 1947

John Dewey
1859 - 1952

Seymour Papert
1928 - 2016

Figure 1. My three intellectual heroes.

5.3 The search for a curriculum structure

Whitehead's power and beauty of ideas and Dewey's experience of the artist both emphasize the richness of the learning experience and the importance of the training of the mind. I have argued (Taylor 2018) that the writings of both these philosophers have a lot to offer us today. Raymond (2018) agrees with this but suggests that these ideas might have had little effect on classroom practice.

I start with Whitehead. His Rhythm of Education (1929: Chapter II) effectively provides a structure for the curricula of all disciplines. Here he identifies three stages of learning: Romance, Precision and Generalization. To some extent, our learning proceeds through these three stages in order, such that, roughly speaking, the child is dominated by Romance, the youth by Precision, and the adult by Generalization. In practice, however, the stages cycle continuously like eddies in the fast-flowing stream of life (and indeed at different times we can all be children or adults).

The first stage, of Romance, is one of ferment, novelty and mystery, of hidden possibilities and barely justifiable leaps. This stage, in its fullness, motivates the second stage, of Precision, in which we strive for comprehension and mastery—ideas must be tamed and organized, requiring care, honesty and restraint. Finally, the third stage, of Generalization, is essentially a return to Romance, but now with the technique acquired at stage two. Our ideas have new power because we have harnessed them. The great fruit of this ultimate stage of learning is wisdom: the capacity to handle knowledge. The central point that Whitehead makes is that the discipline of stage two must not be imposed until the fullness of stage one has properly prepared the student. Failing that, the knowledge that is obtained will be inert and ineffective.

This "rhythm" sets a structure for the entire 12 years of schooling, one which will hopefully sustain us for the remaining years of our learning. For each particular course and indeed for each learning hour, it provides a ritual that we too often fail to observe. I find that it makes a great difference if, when planning a lecture, I remind myself of the precedence of Romance. Certainly Whitehead's rhythm lays to rest that ridiculous conflict between discovery and basics; the first most often provides the Romance, the second the Precision.

Moving on to John Dewey, his search for a structure is encapsulated in the title "The need of a theory of experience." of Chapter 2 of his 1938 essay *Experience & Education*:

> I assume that amid all uncertainties there is one permanent frame of reference: namely, the organic connection between education and personal experience." (1938, page 8).

That "frame of reference" is what defines the structure of Dewey's encounter with education. He had of course already, in 1934, developed that theory in the powerful context of the aesthetic. There, his attention was on the audience much more than on the performer, particularly in his insistence that the heart of the aesthetic experience is found in the response of the viewer.

> The word "aesthetic" refers, as we have already noted, to experience as appreciative, perceiving and enjoying. It denotes the consumer's rather than the producer's standpoint. It is Gusto, taste; and, as with cooking, overt skillful action is on the side of the cook who prepares, while taste is on the side of the consumer, as in gardening there is a distinction between the gardener who plants and tills and the householder who enjoys the finished product (Dewey, 1934, p. 37).

In fact too much emphasis on the "finished product" can detract from the experience. The opening paragraph of *Art and Experience* emphasizes this:

> In common conception, the work of art is often identified with the building, book, painting, or statue in its existence apart from human experience. Since the actual work of art is what the product does with and in experience, the result is not favorable to understanding. In addition, the very perfection of some of these products, the prestige they possess because of a long history of unquestioned admiration, creates conventions that get in the way of fresh insight. When an art product once attains classic status, it somehow becomes isolated from the human conditions under which it was brought into being and from the human consequences it engenders in actual life-experience. (Dewey 1934, p. 1)

Some time ago it was not uncommon the hear teachers proudly proclaim: "I don't teach math; I teach students." I thought at the time that this was a bit silly because of course, we do both. But I'm guessing that the purpose of the phrase was effectively to reinforce Dewey's important insight.

This then brings us to what Dewey calls the central problem of an education based upon experience: "to select the kind of present experiences that can live fruitfully and creatively in subsequent experiences." (1938, p. 9).

The conclusions he draws from that are, on the whole, well understood today, for example that meaning comes only from the present experience of the student, and that subject matter earned in isolation, put, as it were, in a water-tight compartment to be opened only at the time of the exam, contributes nothing to the student's future life. But although these conclusions are well understood they are widely ignored. When I am working in a high-school classroom I put the students in groups either at tables or (preferably) standing at white or black boards and I evaluate the quality of the problem in part on signs of an engaging and even intense experience.

Finally I add one more layer to this search for the right structure, and that emerges from Seymour Papert's idea of a project as a significant activity that provides meaning to the student's life.

> The important difference between the work of a child in an elementary mathematics class and that of a mathematician is not in the subject matter (old fashioned numbers versus groups or categories or whatever) but in the fact that the mathematician is creatively engaged in the pursuit of a personally meaningful project. In this respect a child's work in an art class is often close to that of a grown-up artist. (Papert 1972, p. 249).

More recently, Jo Boaler makes the same point comparing mathematics to language studies:

> When we ask students what math is, they will typically give descriptions that are very different from those given by experts in the field. Students will typically say it is a subject of calculations, procedures, or rules. But when we ask mathematician what math is, they

> will say it is the study of patterns that is an aesthetic, creative, and beautiful subject. Why are these descriptions so different? When we ask students of English literature what the subject is, they do not give descriptions that are markedly different from what professors of English literature would say (Boaler, 2016, p. 21-22).

In effect this is an argument by analogy that at the school level, we should be teaching the mathematics that mathematicians do (Taylor 2018). I draw from that idea when I find myself constructing a new high school problem. If, when I am writing it up, I, as a mathematician, feel the life and energy waning, that's a signal the problem might not after all be right. On the other hand, if the excitement builds, I feel I must be on the right track.

A "project" for Papert is necessarily a sustained endeavour, and that has a number of consequences:

> This *project-oriented* approach contrasts with the *problem* approach of most mathematics teaching: a bad feature of the typical problem is that the child does not stay with it long enough to benefit much from success or from failure. Along with time-scale goes structure. A project is long enough to have recognizable phases—such as planning, choosing a strategy of attempting a very simple case first, finding the simple solution, *debugging it* and so on. And if the time scale is long enough, and the structures are clear enough, the child can develop a vocabulary for articulate discussion of the process of working towards his goals (Papert 1972, p. 251).

The last idea of this remarkable paragraph is worth highlighting. Math students often have trouble talking about the subject they are studying; they lose the big picture, if they ever had it, and they get lost in the details. Papert suggests that a habit of sustained engagement can foster discussion at the structural level—if the structure is rich, there is more to talk about.

Barabe and Proulx (2017 p 26) make the important point that Papert's projects emphasize *doing* more than *knowing* and thereby give the students something much more powerful than mathematical knowledge and that is what Papert calls "mathematical ways of thinking." That's really another way of saying that we should be teaching the mathematics that mathematician do.

For me this project structure has the power to give us a natural realization of the structures put forward by Whitehead and Dewey. When our curriculum planning is on the level of the project, we seldom need to search for Romance; it is typically already in place as an organic component of the process. In the same way, Dewey's "experience" is typically an integral part of the activity generated by the problem. I find that when I am considering whether or not a problem passes the bar of admission to my classroom, I pay early attention to the student experience (Dewey's "consumer"), looking for aspects such as surprise (Gadanidis et. al 2016), wonder (Sinclair and Watson 2001), flow (Liljedahl 2018), beauty (Sinclair 2006), low floor, high ceiling (Gadanidis et. al. 2016 p. 236, Boaler 2016 p. 115).

And of course a project-oriented curriculum structure is much more creative, challenging and even "humanizing" for the teacher; it can nurture her development as an artist.

5.4 Towards a project-oriented curriculum

Time to sum up and put things together. The more I reflect on the present reality of high school math, the more of a disaster it seems. That's strong language but it's what comes to mind when I think of the students. Quite simply, they deserve better––they deserve the real thing. That simple truth strikes me most forcefully when I go into the classroom and work with them. For the most part, they are ready to work and more importantly, they are ready to play.

Of course, as things stand at present, most of them feel that what they are getting in the classroom is what mathematics is; indeed they simply don't know what they are missing. More than once, after 75 minutes in the classroom, I get the comment, "why isn't math always like this?" I do note that, back in the 50's, we did at least encounter the grandeur of the subject, as in Grade 10 we had a full-year course in Euclidean geometry.

So what *are* they missing?—the best way to answer that is to observe that mathematics is the study of structure, and that high school math currently offers no identifiable structures of any sophistication. Papert's projects offer us a way towards a curriculum with genuine mathematics. But how do we get there?

There are difficulties. First of all projects are harder to work with and often require a level of mathematical and pedagogical experience that many teachers do not yet have. And there is the question of time. The activities take time and patience, and teachers often feel that the job of building a proper technical foundation for their students already takes almost all of the available class hours. And finally, because my visit is effectively an intervention, the activities can seem disconnected and even contrived. I will discuss each of these factors.

5.4.1 The technical skills

They are important; we can't do mathematics without them. But if we assemble ahead of time all the ones we think we might need, for example to do calculus, the basket will be too heavy and will divert us from the real goal. To work and play effectively, we need to travel light, and that requires putting that basket aside and having the simple faith that the activities we choose will be comprehensive enough to look after the student's future technical needs. Those who worry that the students might miss some critical skills should spend some time in a first-year university calculus course and find out that many of the skills that were "taught" in high school were not in fact learned in any effective way. Skills need meaningful context; the more powerful the context, the more solid the skill.

What *is* important is that students learn how to master skills. That's well understood by students who play guitar or basketball; they simply have to realize that the same principles apply to mathematics. This idea works so seamlessly in music and sports because they in fact have that powerful context. Well, mathematics has an equally powerful context to offer, but it's one that few students have ever encountered.

The other thing to notice is that universities, professional programs and employers are increasingly emphasizing a new level of what are often called "secondary" skills, sometimes called the "C-words"—care, creativity, critical thinking, communication and collaboration. A project-based curriculum can often relate more naturally to these.

5.4.2 Teacher preparation

Even experienced teachers find it a challenge to work with investigative activities. First of all there are usually different ways tackle the problems and it helps to be able to anticipate these. That takes more in the way of preparation time and, often, mathematical knowledge as well. And there are balances to be struck—between giving the students ideas and letting them find avenues on their own, between keeping the class together and giving the faster students questions on the side, between individual work and collaboration within groups.

A project-oriented curriculum can be an enormous challenge for teacher candidates. My colleagues in Faculties of Education well realize that this is an increasingly important part of their job, but there is only so much they can do. The simple fact is that most of our learning about how to teach happens when we ourselves are being taught and most of today's fledgling teachers have spent too little time in their own mathematics learning exploring and investigating. I will mention three phases of that experience. One of these is their school experience and that's not surprising as that is of course exactly what we are working to change. Another is the time they spend out of school and there is evidence that the technological and media explosion has seduced many of them away from much of that experience. The third is their undergraduate learning and that is an experience that many of the readers of this volume have some control over. I am definitely not happy with the nature of most of the teaching in undergraduate math courses in North-American universities, particularly in the "service" courses, and those are often the courses taken by future math teachers. These courses need to purvey less in the way of mathematical knowledge and put much more emphasis on inquiry and mathematical thinking. Students who might actually need considerable mathematical knowledge typically already know that this is the case and respond accordingly.

5.4.3 What mathematics?

I want to briefly return to this question of the dominant place the study of functions plays in the senior school curriculum, certainly in North America. I have observed that the cause of this is almost certainly the role of calculus as the default math course in first-year university and college. Now whether that remains the case or not, my belief is that the current introductory calculus course offered in the senior school curriculum is not the right preparation. It is technical in nature and is very much oriented towards the transfer of mathematical knowledge, with little attention given to mathematical thinking. It also gives the students the misleading impression that they have already covered much of the first semester of university calculus. I would prefer a course with a theme of modeling and optimization, using many different approaches, analytical, geometric and graphical. It would not follow the logical technical development of the subject, leaving that for university, but would still remain true to the *ideas* of calculus. The few technical pieces such as the arithmetic laws of the derivative could be quickly covered and then employed "in action," thus remaining true to Whitehead's Romance and Dewey's present experience.

I illustrate these remarks with two examples taken from my own body of work (Taylor 2016). Example 1 is a model for the speed at which a car should be driven to minimize the cost of gas.

5.4.4 Example 1. Gas consumption for optimal driving speed.

We need to start with a graph of gas consumption against speed and there are some simple mainstream kinetic energy principles that lead to a simple equation for this. A senior class that has some acquaintance with Newton's Laws of motion will enjoy the challenge of finding the algebraic form of the gas consumption graph found in Fig 2a). It gives rise to some interesting questions such as why is it expected to be concave-up. For the various components of the problem, we have the choice of working with the formula we have derived and using algebra or even calculus, or working with the geometric form of the graph, or of course both. I will highlight the graphical argument.

To begin we ask for the velocity that minimizes the cost of making a trip of a fixed distance. Now the vertical axis z has units in litres consumed per hour at any fixed speed v. But to use least gas over a given distance, we want to minimize litres per km (z/v) and that requires us to minimize the slope of a secant line drawn from the origin to the graph. This occurs when the secant is tangent to the graph, and the optimal speed in this case (Fig 2b) is 50 km/h. This is considerably slower than we typically drive on the highway and the reason for this of course is that we put a value on our time; to account for that, what we really need to minimize is the sum of gas cost and the effective wage we are paying ourselves. This sum is minimized with an elegant generalization of the secant construction of Fig 2b. Putting the cost of 6 litres of gas as the value of an hour of our time (thus with a gas cost of \$1.50/L this would be \$9/h), Fig 2c gives us the reasonable optimal speed of 90 km/h. This is a rich, multifaceted problem that can be tuned and extended in different ways at different grade levels. It certainly earns the status of a Papert project.

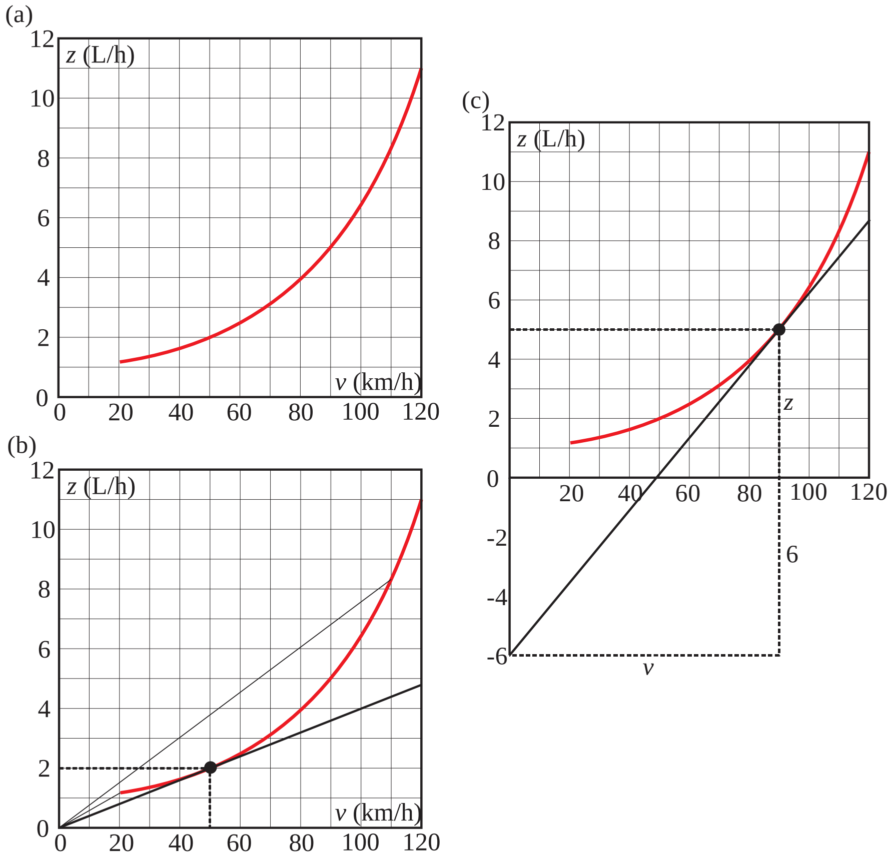

Fig 2. Optimal driving speed. (a) The gas consumption graph derived from energetic considerations. (b)The optimal speed for a trip of fixed distance. (c) the optimal speed that incorporates an effective wage paid to the driver.

5.4.5 Example 2. Counting trains

Some branches of mathematics lend themselves more readily than others to investigation and what is called "mathematical thinking." In my experience projects involving discrete structures, geometry, simple probability, strategic thinking (games) are more accessible to students and more naturally investigative than is the study of functions. I suggested earlier that the time might have come for us to seriously consider a change in the mix of the mathematical areas taught in high school and even university. In that regard, I offer a modeling project from discrete math. Discrete structures appear as a topic in most secondary math curricula, but these often focus on applications to financial math and seldom exhibit the structural richness that these topics can offer. In this project the students are challenged to provide proofs for some of the well-known Fibonacci formulae by invoking properties of the structure of trains.

Problem 1. I want to construct a train of total length n units using cars which are either 1 unit long or 2 units long. The question is, for each value of n, how many different trains are there?

For small n we can simply write all the possible trains down. Thus Table 1 shows that there are 8 different trains of length 5. To have some notation, we let t_n denote the number of trains of length n. Thus Table 1(a) shows that $t_5 = 8$ and Table 1(b) gives the results of similar counting exercises.

(a)

Trains of length 5
1-1-1-1-1
1-1-1-2
1-1-2-1
1-2-1-1
2-1-1-1
1-2-2
2-1-2
2-2-1

(b)

Length n of train	**Number of trains t_n**
1	1
2	2
3	3
4	5
5	8
6	13

Table 1. (a) The 8 trains of length 5. (b) The train numbers for small n.

How do we handle large n? There is in fact a standard combinatorial approach--look at the different possibilities for the number of 2-cars, and count the number of arrangements of each. But there is a much simpler argument using the power of recursive thinking.

We start by collecting some data—actually counting the different trains for small values of n by listing the possibilities (Table 1).

Many students will recognize the pattern in Table 1(b) as the well-known Fibonacci sequence. The "law" of these numbers is that each term is the sum of the two preceding terms. Once they see this pattern, they figure that the problem is solved. For example, the number of trains of of length 7 will be 8+13=21, etc. But can we be sure of that? Do we know for sure that the pattern continues? That’s our first problem.

Let me point out that there's more here than a question of certainty. If this really does hold, one would think there ought to be a simple elegant argument for it (after all there's nothing very complex going on here) and it is that “expectation” of elegance that motivates the mathematician.

Establishing the recursion. Our task is to convince ourselves that the simple sum rule should hold for the train sequence. As a specific example, take the equation:

$$t_7 = t_6 + t_5$$

Find an argument that the number of 7-trains *has* to be the sum of the number of 6-trains and the number of 5-trains.

I give this question to the class, but they’ve never seen anything like it before and hardly know where to begin. How could such an argument ever be constructed? I give them a hint—well it’s more than a hint, it’s a simple but powerful idea that will serve them well in all the remaining problems we will look at.

The equation asks you to show that one quantity (t_7) is the sum of two other quantities (t_6 and t_5). Now all three numbers are the sizes of sets of objects with a particular structure. Maybe there’s a natural way (using the structure) of partitioning the objects in the t_7-set into two types that corresponding naturally to the objects in the two other sets (t_6 and t_5).

For example imagine that you are atop the CN Tower looking down at all 21 trains of length 7, each of which has an engineer. Think of an instruction you can give the engineers: “if your train has the following property drive it to the east, and if it doesn’t drive it to the west” such that there is a natural 1-1 correspondence between the trains that go east and all the trains of length 6, and between the trains that go west and all the trains of length 5.

This is the hint I give the students but many of them have more trouble with it than I have expected. That convinces me more than ever that this is the sort of analysis that needs to appear earlier in their lives.

Here’s the argument (Fig 3). The trains of length 7 are of two kinds: those that begin with a 1-car and those that begin with a 2-car. Now how large is each set? Well it’s clear that there are t_6 trains in the first (the rest of the train can be any train of length 6) and t_5 in the second (the rest of the train has length 5). So t_7 must be the sum of those two numbers.

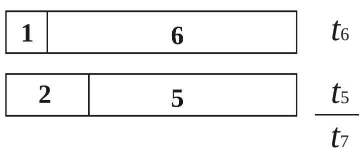

Fig 3. The instruction for the engineers—if your train starts with a 1-car, go east, and if it starts with a 2-car, go west.

A number of students come up with this argument but in a slightly looser form. They say: put a 1-car in front of all the trains of length 6 and put a 2-car in front of all the trains of length 5—in both cases you get a train of length 7. That's correct but to get the sum formula you do have to verify (or point out) that you get *every* train of length 7 with one or the other of these constructions and no train of length 7 will get counted twice.

It's clear that this argument is quite general, and could be used to show that the number of trains of length 8 is equal to the sum of the number of trains of length 7 and the number of trains of length 6, etc. So the additive rule always holds.

$$t_{n+1} = t_n + t_{n-1}$$

We deduce from this that the train numbers are given by the Fibonacci numbers, and we can therefore continue the table as far as we wish (Table 2a). For example, without doing any counting, we can be sure that the number of trains of length 12 is 233.

(a)

n	t_n
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233

(b)

$t_n{}^2$	
1	
	2
1	
	5
4	
	13
9	
	34
25	
	89
64	
	233
169	

Table 2 (a) The train numbers. It is often mathematically convenient to start the count at n=0. Can we make "train" sense of this? Perhaps—there's only one train of length 0 and that's the empty train. (b) Sums of squares of neighbouring Fibonacci numbers.

The unexpected power of the trains-numbers. We have established a correspondence between the train numbers and the Fibonacci sequence. This is an elegant mathematical result. But it is also unexpectedly powerful. Here's why.

The Fibonacci numbers possess many wonderful arithmetic properties, but most of these are not so easy to prove. Here's a remarkable idea. Take any of these and interpret it in terms of trains. Perhaps the structure of trains will give us a way to establish the property and even "see" why it ought to be true. Here's an example.

Problem 2: Sums of squares. Take two consecutive Fibonacci numbers and add their squares. It appears that we always get a Fibonacci number (Table 2b). For example:

$$5^2 + 8^2 = 89$$

This is a fascinating property and it's not at all easy to see why it should be true or to get any kind of intuition for it. [If you don't believe me, give it a try.] So we ask:

Can we find a train-theoretic argument for this property?

Let's look for such an argument for the special case $n = 5$:

$$t_{10} = {t_5}^2 + {t_4}^2$$

Following our previous idea, we look for a natural partition of the trains of length 10 into two disjoint classes with ${t_5}^2$ trains in the first class and ${t_4}^2$ trains in the second. A clue comes from noting that the subscripts 5 and 4 have the status of being roughly half of 10—this suggests that the classification ought to be based on something like "cutting the trains of length 10 in half." *Can you take it from there?*

Well here's an argument I often get from students. Take a train of length 10. Its first half is a train of length 5 and its second half is a train of length 5. There are t_5 possibilities for the first half, and for each of these there are t_5 possibilities for the second half. So that's a total of ${t_5}^2$ possibilities. We conclude that there are ${t_5}^2$ trains of length 10.

Except there aren't—there are evidently ${t_5}^2 + {t_4}^2$ such trains. So what went wrong?

It doesn't take long to see the problem—not all trains can be cut in half. What we have effectively argued is that there are ${t_5}^2$ trains of length 10 *that can be cut in half.* And from the formula, we guess there must be ${t_4}^2$ trains of length 10 that *cannot* be cut in half. So what stops a train from being cut in half?—if there's a 2-car right in the middle! In that case, to cut the train in half you'd have to cut the 2-car in half. Okay—how many trains of length 10 are there that have a 2-car in the middle? Well the part in front of the 2-car is a train of length 4 and the part behind the 2-car is a train of length 4 and that's a total of ${t_4}^2$ possibilities (Fig. 4). Now that's an argument of great beauty!

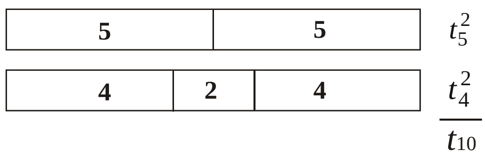

Fig 4. The "trains proof" of the sum-of-squares property for the Fibonacci numbers.

There are many more Fibonacci examples of this form of argument. It can also be used to illuminate some of the remarkable relationships between the Fibonacci numbers and Pascal's triangle. An example is found in Fig 5.

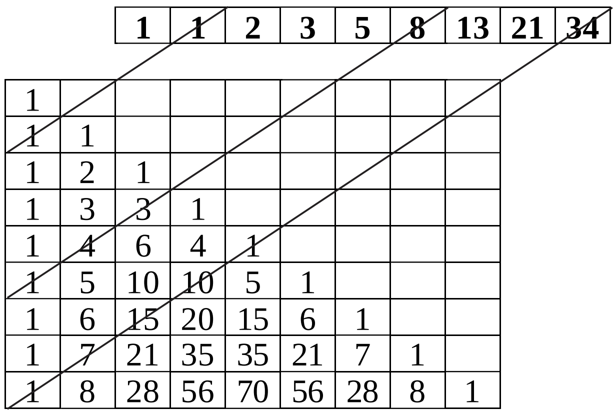

Fig 5. One of the many remarkable relationships between Pascal's triangle and the Fibonacci numbers. Can you find a "trains" argument?

It is interesting to note that in these problems, the train numbers serve as a "model" of the Fibonacci numbers, but the modeling in this case is the reverse of what normally happens. Typically we have a real-world situation (e.g. minimizing gas consumption) and we find abstract mathematical equations to describe it and establish new properties. But here we are starting with an abstract entity, the Fibonacci numbers, with a number of observed properties, and we are using a real-world structure (trains) to establish these properties. Fascinating.

5.5 Conclusions.

There is much current interest in mathematics education at the elementary level, and this is also the case at the tertiary level. But there is not so much at the secondary level. It is generally agreed that the job of high school is to prepare students for college and university and for the most part that's about technical proficiency and, in that regard, the current curriculum is doing the best it can. In my view, this is far from being the case. It is true that students need a good level of technical mastery, but here is Whitehead's commentary on that:

> The mind is an instrument; you first sharpen it, and then use it… Now there is just enough truth in this answer to have made it live through the ages. But for all its half-truth, it embodies a radical error which bids fair to stifle the genius of the modern world… The mind is never passive; it is a perpetual activity, delicate, receptive, responsive to stimulus.

> You cannot postpone its life until you have sharpened it… There is only one subject-matter for education, and that is Life in all its manifestations. (Whitehead 1929, p. 6).

We can look at the math curriculum through many lenses. One of these is the subject matter that is taught and I have discussed that above. Another has to do with the level of sophistication and that is a major theme of this chapter, particularly in terms of structural sophistication. A third has to do with the pedagogical approach and I certainly come down on the side of an investigative curriculum. And I have argued that this can work only with a curriculum structure that puts technical considerations aside and focuses on Dewey's experience and Papert's projects.

There is another aspect of this that I want to emphasize in closing and that is closely related to the concept of integrity. Put yourself in the role of the teacher who goes into the same classroom each day. What you do there with your students needs to reflect and reaffirm your human nature; for your students it is who you are, and it is also what mathematics is. There needs to be a unity or wholeness about that and a curriculum that supports and nurtures that has what I want to call "integrity." On a philosophical level, we can see aspects of this in the ideas of Whitehead, Dewey and Papert, that in a real sense they are all talking about the nature of the human experience. Now move to the level of the students, sitting in formal rows, or moving chaotically among neighbouring whiteboards; what are they noticing? Of course they are attending to the mathematics, the more so if it is engaging. But a significant slice of their attention is surely focused on character and mood and unity, indeed on the *integrity* of the experience. That will inform not only their view of mathematics, but also their evaluation of ideas, of learning, of what school is all about, and most importantly, their allegiance to their teacher.

Given that, it is more important than ever that what happens in the classroom be real mathematics, the mathematics that mathematicians do. Of course, no matter what grade we are working with, the mathematics we are considering will likely have to be tamed or engineered to fit inside our classroom. Having said that, I have often been amazed at what can be done with a sophisticated activity and at how well the students are able to step up to the plate.

Mind you, when I say "real mathematics," I don't mean that it has to be extracted from a research paper or a 4^{th}-year seminar. It simply has to be something that interests and even delights a mathematician, that it has him or her, at the first opportunity, whipping out a pencil and sitting down to play.

For example consider the equation.

$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$$

No mathematician I know can resist that equation, and I and my teacher colleagues have gone a long way with it in a number of grade 9 classes, introducing the students to the power of abstraction. Even more intriguing (and somewhat more advanced) is the equation

$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}}$$

Of course in both cases, the problem is to find other examples with the same structure. One of my projects (Taylor 2016) is built around a collection of such equations.

Peter Liljedahl (2018) uses the word “flow” to describe the way in which a good problem or activity moves the student seamlessly along from one stage to another. I look for that when I am working with a class. When this happens the energy is palpable and it can be a challenge keeping the class together. This brings to mind a wonderful passage of John Dewey.

> Experience in this vital sense is defined by those situations and episodes that we spontaneously refer to as being "real experiences"; those things of which we say in recalling them, "that was an experience." It may have been something of tremendous importance--a quarrel with one who was once an intimate, a catastrophe finally averted by a hair's breadth. Or it may have been something that in comparison was slight--and which perhaps because of its very slightness illustrates all the better what it is to be an experience. There is that meal in a Paris restaurant of which one says "that was an experience." It stands out as an enduring memorial of what food may be. Then there is that storm one went through in crossing the Atlantic—the storm that seemed in its fury, as it was experienced, to sum up in itself all that a storm can be, complete in itself, standing out because marked out from what went before and what came after…In such experiences, every successive part flows freely, without seam and without unfilled blanks, into what ensues . (Dewey 1934 p. 43).

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