Afterward: Math and the Future of the Planet

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In the first paragraph of her introduction, Ann talks about you as a teacher and where your main focus must be. She makes the point that the transition from being a learner to a teacher requires a deeper understanding of the workings of the subject, and in the book, she shows you how to "play" with the ideas and techniques to achieve that understanding.

In this short essay I will argue that you can and should use that understanding to raise the level of sophistication of the mathematics you bring into the classroom. By doing that you will provide a richer mathematical experience not only for your students, but for you as the teacher.

There has been considerable discussion around the need for this richer experience for our students. In a 1996 article, Al Cuocco et. al. propose that

> For generations, high school students have studied something in school that has been *called* mathematics but has very little to do with the way mathematics is created or applied outside of school. One reason for this has been a view of curriculum in which mathematics courses are seen as mechanisms for communicating established results and methods-for preparing students for life after school by giving them a bag of facts. (Cuocco, Goldenberg and Mark 1996, 375)

And Jo Boaler in her 2016 book *Mathematical Mindsets* asserts:

> There are other indications that math is different from all other subjects. When we ask students what math is, they will typically give descriptions that are very different from those given by experts in the field. Students will often say it is a subject of calculations, procedures, or rules. But when we ask mathematicians what math is, they will say it is the study of patterns and is an aesthetic, creative, and beautiful subject. Why are these descriptions so different? When we ask students of English literature what the subject is, they do not give descriptions that are markedly different from what professors of English literature would say (21-22).

Boaler's comparison to the English curriculum is of substantial interest to me and I will return to that.
In 2018 I wrote a paper arguing that we should "teach the mathematics of mathematicians":

> I take it as understood that mathematics is a central discipline in the school curriculum. As a consequence, it is important that the teaching and learning of mathematics be held to a high standard. However it has frequently been remarked that in our K-12 school system, this high standard is not generally met. There are two aspects of this. One focuses on student knowledge and performance—students seem to have little knowledge of the subject and they can *do* very little with what they do have. The other concerns the student experience. Mathematics is a subject full of wonder and beauty, but students in secondary school rarely experience that. (Taylor 2018,1)

And George Gadanidis of Western University has argued:

> Teaching mathematics is not just about how we teach and how students learn. It is also about what mathematics we bring to the classroom. We have become very good at pedagogy. We can walk into a classroom, or reflect on our own teaching, and recognize what works well pedagogically and what does not. We also need to be able to recognize and be concerned about the quality of the mathematics we bring to the classroom.

Finally Ann refers to this need to go "beyond pedagogy" when she writes:

> [This] is a book on *mathematics*, not pedagogy. It does not focus on lesson design, questioning techniques, assessment, and so on. But what it may do, is allow and support you to think more deeply about the mathematical ideas in such a way that these pedagogical aspects become much more effective.

Okay. There is a practical question that arises right away. Is it really possible to find that deeper, more authentic “mathematics of mathematicians” that

- our students are ready for
- fits the mandated curriculum.

These are both important issues.

Readiness—what our students can do

I am often amazed at the level of sophistication my students can rise to, if they have a problem that lifts them up. Ann explicitly refers to this in her introduction:

> On the other hand, experiences of meaning, ownership, action, involvement, and problem solving can actually motivate students to want to figure things out.

Rabbitmath is a Canadian project based at Queen’s University and University of Ottawa with the objective of reimagining the high-school curriculum (see the Rabbitmath website, accessed August 28, 2022). We recently ran a RabbitMath camp with students from grades 7-10. We gave them problems that would normally be considered too sophisticated for those grade levels, and we were impressed with how well they performed. I will provide a few of our Desmos problems below.

I will say that one of the things that makes us cautious about giving students problems that they are “not ready for” is the apparent importance of the prerequisite structure in mathematics, arguably of greater significant in math than in any other discipline. I have read and thought about this long and hard and am convinced it is not what it seems to be.

The belief that mathematics needs to be taught in a logical sequence (that you need to understand A before learning B) has been around for a long time. In my view, and that of many others, it has ripped the soul out of the subject. Whitehead calls it a “half-truth that embodies a radical error which bids fair to stifle the genius of the modern world.” (Whitehead 1926, 6).

Because of such destructive effects it is important to try to understand why this prerequisite idea has had such power. One reason of course is that, being a half-truth, it *does* have an element of truth. For the most part that truth comes from the logical character of the subject articulated as a finished product. But mathematics the finished product and mathematics being learned are quite different animals—one is an owl, the other a rabbit. Students do not learn math in sequence, but as a complex and even chaotic network that is constantly being updated and reordered. And of course the activity of mathematics research proceeds in that chaotic manner as well.

This is a guiding principle in RabbitMath. While we do pay attention to the overall mathematical content of the curriculum, at each moment, we follow the call of the structure we are studying and in any particular course, we allow ourselves to work with mathematical ideas and techniques from many grade levels. Our experience is that solving problems they are not ready for is a real energizing experience for our students.

The fit to the given curriculum

This is the more challenging and interesting part of the question. I am convinced that our conception of “fit” is far too narrow. I am going to talk about the English curriculum, but I start by recounting my experience teaching a wonderful course with no set curriculum—a real luxury.

For some 20 years, I co-taught a course called mathematics and poetry. It had 60 students in 3rd or 4th year drawn from all over the university and we met every Wednesday evening for 3 hours. There were two of us, an English prof (first Bill, and after Bill died, Maggie) and me. We met in the evening (7-10) once a week and for the first half of the class we played with a poem, and for the second half we played with a math problem. There was no set curriculum so we were able to choose the very best “works of art.” In each half, in a discussion-oriented format, the students did the same thing with the math that they had done with the poetry; they took the work of art apart in different ways, then they put it back together, sometimes in a new way, with new allegiance and new understanding. I grew to love the poems we used and they remain my favorites today. And I remember my sense of anticipation walking into class each Wednesday evening with a new math problem—*This is pure gold; they are going to love this.*

In fact, I believe they did, and when, from time to time, I meet alumni, *Math and Poetry* is the course they remember. And that's not only because the poems and the math problems were so interesting; it was also the wonderful sense of freedom that we all felt, the students as well as the instructors. That freedom was what allowed me to choose math problems that were just right for the students, indeed I should say, for the students *and* for the mathematics.

One of the students in the course was Adam Brown, at the time a math major in 3rd year. Adam subsequently did a Master's degree with me and is now a secondary math teacher in Toronto. He visited me last week, and described his first evening in class. I was telling the students about the course and I promised that the poems and the math problems would be accessible to everyone, and challenging for all. And Adam said he looked around at the huge mix of disciplines in the class (we had introduced ourselves) and he said to himself: no way—how could the math problems be challenging for me, and still be accessible for students who only had grade 12 math?

Well, he added, I was surprised to discover that this was true of both the poetry and the math. Adam found them both to be challenging. Furthermore, he recalled, the key insights for the poems sometimes came from the math and physics folks and the key insights for the math problems sometimes came from students in the arts.

And now, later in life, I am trying to find that sense of freedom in all my courses. In addition, since for the past few years much of my time has been spent constructing new math problems, I want the school mathematics classroom to have that same spirit. I want the teachers' principal objective to be to walk into class with a problem that they love and that their students will want to play with and talk about and take home and play with some more and maybe even grow to love themselves. For me, "play" means building and designing and creating cool things. Just like art class.

That comparison with art class comes from Seymour Papert. Seymour, along with A.N. Whitehead and John Dewey are my three Math Ed mentors. There have been some great articles written over the past decade, but, for me anyway, the fundamental ideas were already set down by those three "philosophers."

Math and English

To understand all this better, I have found it illuminating to study the Ontario (Canada) Curriculum document for English. It has the same general format as the Mathematics document except the Specific Expectations are not content oriented, rather they relate directly back to the Process Expectations in the first part of the document. Another way to say this is that the development of the English student is largely thought of, *not in terms of what they know, but in terms of what they can do* (or say, or write). For example, a grade 9 student could discuss any of the poems from our math and poetry course, but it would be at a different level of sophistication from the response of senior undergraduates. The novel *To Kill a Mockingbird* is studied in grade 9 as well as in a graduate English seminar, but at a different level. The point is that in literature, and indeed quite generally in the arts and the humanities, students are able to engage with works of art that are significantly above their level of sophistication. In fact. that's a critical part of their academic development.

In short, when our focus switches from *what they know* to *what they can do*, the grade level no longer plays the huge role it used to. To be clear they do not have the knowledge that they will gain in the senior grades, but we show them how to do things (or they discover that through exploration), and then they are free to run.

Let me emphasize this. If I happen to be working with a bunch of grade 8 students using a problem that looks like it belongs in grade 11, I am *not* giving them the grade 11 knowledge that belongs to the problem. In fact I am not playing the knowledge game at all; I am playing the process game. I am focusing not on what they know but on what they can do.

That's of course what happens in English class. When I am working with *To Kill a Mockingbird* I do not worry that I might be trespassing on a possible future encounter with that book. Sophistication is about structural richness—there are almost always new insights to be had.

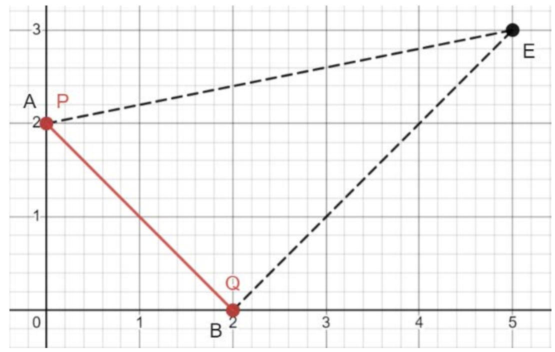
I want the structure of the math curriculum to be more like that of the English curriculum in precisely that sense. Our poems are the problems; our novels are the rich structures that emerge when two branches of mathematics come together.

Indeed, over the past few years, every time a question or comment has come my way about the math curriculum, my response has been to ask the same question about the English curriculum and I unfailingly find the comparison insightful. It has helped me to uncover some hidden assumptions we make about what we need to teach and how we need to teach it.

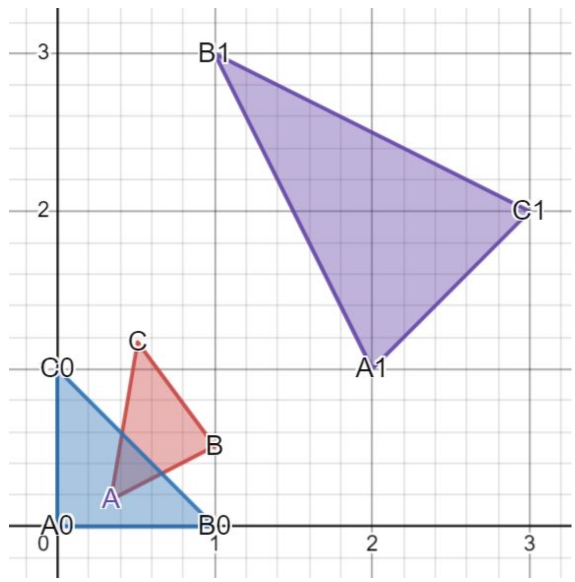
Okay, knowing and doing. I'm going to end with that. In education it does seems that our focus is on knowledge: what does the student *know*? More and more I've wanted to focus on what can my students *do*? That small shift can transform the classroom, and for me it holds the key to my freedom as a teacher.

Example 1. Some Desmos activities from the Grade 7-10 RabbitMath camp

A big theme at the intermediate level is linear change so we've been showing students how to connect two points with a line and to move a third point along the line. That allows them to build things like this:
https://www.desmos.com/calculator/xawvuldp9f
When they learn how to move a point along a line they meet the important idea of a parameter.



Once we introduce them to the Desmos polygon function, they can construct things like this. Here we have:
an initial triangle $A_0B_0C_0$
a final triangle $A_1B_1C_1$
The points *A*, *B* and *C* move between the A-points, the B-points and the C-points in a straight line at constant speed, completing the journey in unit time.
https://www.desmos.com/calculator/zvtdvxcwuo



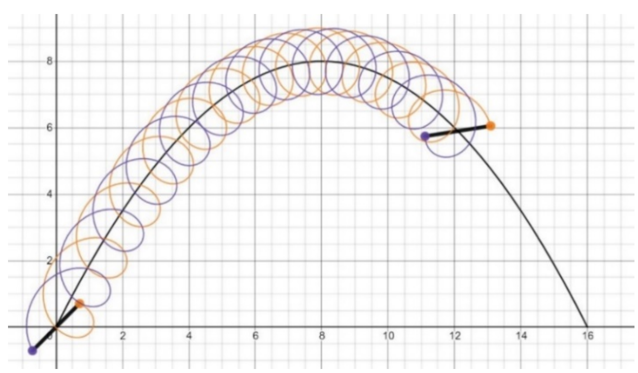
We can ask some interesting questions. Here are a couple that challenged the grade 9s.

(a) Are there any times at which the triangle ABC is a straight line?
(b) The initial and the final triangles are both isosceles. Does this happen again for ABC?

Answers: (a) $T = \frac{\sqrt{3}-1}{2} \approx 0.366$, (b) $T = \frac{4}{7} \approx 0.571$.

All good except once we let the students loose on Desmos, they start to want to do things they are "not supposed to do." One of these is about moving points, not along a line, but around a circle. Hmmm. Does that mean we'd need to teach them some trig?!

No, it does not. That's the old way of thinking; that's about *knowledge*. All they need is to be told how to *do* it. And to be fair, that's all they asked of me—*how do you do it?*
"Okay. Gather 'round children and let me show you these two operators called sin and cos."
And in no time at all they were able to move the centre of the circle and construct things like this.
https://www.desmos.com/calculator/6adsnodwvl



Example 2. Exponential growth from the Grade 7-10 RabbitMath camp

I'm going to start with an interesting exchange between Ann and me over what I might do in this *Afterward*. I am a mathematical biologist by training and Ann rightly assumed that I might want to feature the capacity of mathematics to give us a better understanding of global challenges such as disease and climate change. Indeed, I have to agree that over the past few decades, mathematics has made extraordinary contributions towards our understanding of the natural world.

But these days I am actually more interested in another remarkable capacity of mathematics and that is to help us flourish as human beings, an idea that was recently elaborated in the book *Mathematics for Human Flourishing* by the mathematician, Francis Su.

> I want us as a mathematical community to move forward in a different way. It may require us to change our view of who should be doing mathematics and how we should teach it. But this way will be no less rigorous and no less demanding of our students. And yet it will draw more people into mathematics because they will see how mathematics connects to their deepest human desires. So if you asked me: Why do mathematics? I would say: Mathematics helps people flourish. (Su 2017, p 484-485).

For me that is just as extraordinary a contribution of mathematics as is its capacity to build models of the biosphere.

Anyway that connects very closely with my love of the artistic dimensions of the discipline and that's what I want more and more to see in the classroom at all levels.

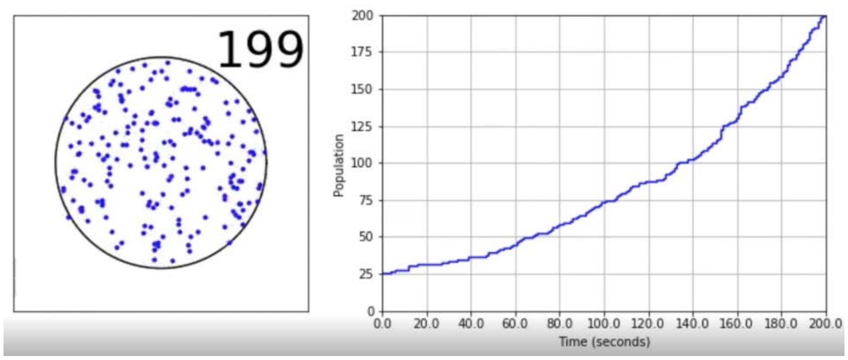
Now on to exponential growth and the first thing to say is that it really is the central mathematical characteristic of the natural world. The reason for that is simple, biological growth rates tend to be proportional to size, and that is in fact the defining property of exponential growth.

Well, when I ask students in grade 12 or even in first-year university what exponential growth *is*, they most often don't know, even though they have often been studying and working with it for years. They tends to say it is "very fast" or "accelerating" growth, or they might try to say something about *e*. Granted, the folks on CBC (the Canadian Broadcasting Corporation) don't seem to know what exponential growth is either.

Anyway, exponential growth is quite precisely growth in which the growth rate is proportional to size. If you double the size, you double the growth rate. It first makes its appearance as a function in grade 11 but it is already encountered earlier in geometric sequences and interest rates. Anyway, because of its importance and the struggle students often seem to have with it, in RabbitMath we believe that it should be encountered much earlier, and we worked with it in our grade 7-10 camp.

In my own years of working with this topic, I often wished for a richer, more hands-on activity of the process and not long ago, one of my students, Becca, constructed the following animation.

The Lightning Model



To put a scale on this process, we specify the following parameters:

radius of the circle: r = 5 cm
lightning frequency 1 per sec
lightning proximity: δ = 0.5 cm
initial pop N = 25

After the students have watched video, we ask them: is the growth exponential, and if so, what is its equation? It is a challenging exercise, even for first-year university students, but with a bit of guidance, most of the grade 7-10 students got to the answer. That was significant for us as this activity definitely belongs in the Grade 11 curriculum.

Let me again emphasize that we were not teaching them exponential growth with all the usual baggage. We simply asked them two questions. How many new particles are created (on average) with each lightning strike, and then (a bit more sophisticated) how can you use the answer to that question to find a formula for the expected number of particles after n lightning strikes.

Example 3. Rolling the Dice

(a) I roll a fair 6-sided die until I get my first 6. How many rolls on average does that take?
(b) I roll a fair 6-sided die until I get two sixes in a row. How many rolls on average does that take?

This is a very rich activity. Not sure where it belongs, perhaps in Grade 12 *Data Management*. But it's a perfect activity for grade 8, and the grade 7-10 campers loved it. Lots of dice rolling, organizing and plotting the data (using Desmos) to get an empirical estimate of the average and finally using a very "operational" approach to get a recursive equation for the average. A beautiful problem.

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Peter Taylor
Queen's University
peter.taylor@queensu.ca
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