Twenty Problems for 7-10
The RabbitMath Team.
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A mathematical problem is authentic if it leads to questions, ideas and structures that a mathematician would want to play with. Can a mathematician possibly be interested in a problem that is right for grade 8 students? Well, we are mathematicians, and the problems in this collection certainly called out to us!

Knowledge and Experience.
Our problems and activities are, for the most part, rather different from the examples found in the Ontario Ministry documents. The main reason for that is that our *objectives* are in many ways different from those that for many decades have ruled the school mathematics landscape. The mathematics curriculum in western countries is *Knowledge*-based; plotting a careful sequential path through the conceptual and technical knowledge to prepare the students for life and work and post-secondary studies. The RabbitMath curriculum is *Experience*-based, and takes its inspiration from the structure of the curricula in the arts and the humanities

Experience
What our students will need to succeed in the world that we are now (for better and for worse) creating is the capacity to analyze and create complex structures. How do we give them that?––we give them what might be called dynamical systems, structures that change over time. So that time is a big parameter. In discrete time we have sequences of objects and we look for recursive relationships to describe how they evolve. In continuous time, lines and curves become trajectories along which points more and the lines and curves and shapes themselves are changing at the same time. Kids love things that move and change, especially when they are in the driver's seat, with the power to design and build.

The Literature
In fact, for a century now, the literature has been calling for exactly this kind of shift in the nature of the activities or problems that we bring into the school classroom. So why has this not happened? Well, it would seem that we have decided that the kids don't have what are described as "the basics" and they will never be able to do math unless they have those in place, so we need to hit the basics first! There is some truth in that, but it is a dangerous truth, too easy to misunderstand and misapply. And anyway, it's all about knowledge.

Low-Floor High-Ceiling,
An experience-based structure is a wild animal. It uses low-floor high-ceiling problems which blur the distinction among grades. So while we generally follow the Ministry Guidelines, we also pursue many ideas (some from the students) that trespass onto neighbouring grades.

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Parametric Lines
https://www.desmos.com/calculator/fgd0zrrkrs

The theme here is the parametric representation of a line (or curve), If we are animating, this is a much more powerful form to work with (than, say, $y = mx + b$) as the parameter, time, is built right into the equations.

In this animation, P and Q move along the dotted lines bringing the line-segment PQ along with them. A good problem is to find the times at which the PQ line is horizontal, vertical, or of slope 1.
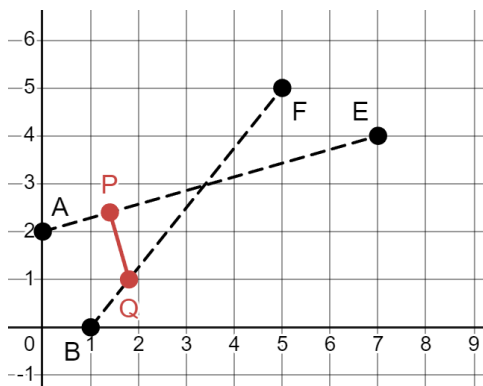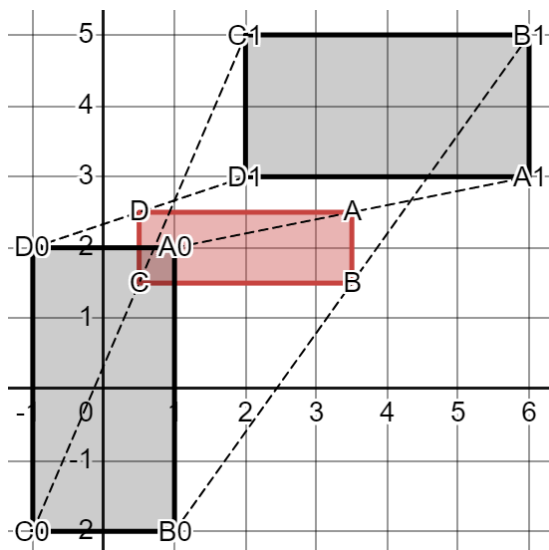
[Parametric Lines Teacher Manual](#)

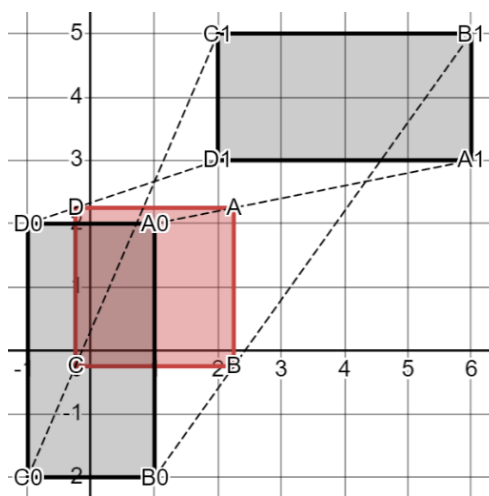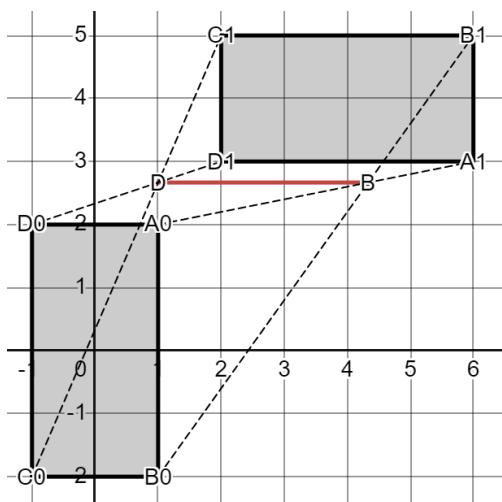**Parametric Rectangle**
https://www.desmos.com/calculator/spnh1xvtyo

Over the time interval $0 \le t \le 1$ the 4×2 rectangle centred at the origin transforms into the 2×4 rectangle centred at $(4, 4)$, such that every point (x, y) moves *in a straight line at constant speed* to its corresponding destination. The diagram at the right is drawn at time $T = 0.5$ so that, for example, A is halfway between A_0 and A_1.

(a) At one point in the journey of the rectangle, it becomes a line segment. At what time does this happen? How long is the line segment?

(b) At one point in the journey, the rectangle becomes a square. At what time does this happen? What is the side length of the square?

[Parametric Rectangles Teacher Manual](#)

This problem is perfect for grade 9 but we find that it also captures the interest of grade 8s.

Parametric Triangles.

Triangle 1
https://www.desmos.com/calculator/mb3pfwffxm

We begin by asking the students to construct:
- an initial triangle $A_0 B_0 C_0$
- a final triangle $A_1 B_1 C_1$

as shown in the diagram. We have them plot and label the vertex points and then use the *polygon* function.

We ask them to construct a red triangle, ABC, whose vertices move in a straight line at constant speed between the corresponding vertices of the initial and the final triangles. For example, A follows a straight line from A_0 to A_1. We choose units so that it takes $T = 1$ seconds to make the journey. The diagram at the right depicts the red triangle at $T = 0.5$. Thus vertex A is ½ of the way along in its journey from A_0 to A_1.
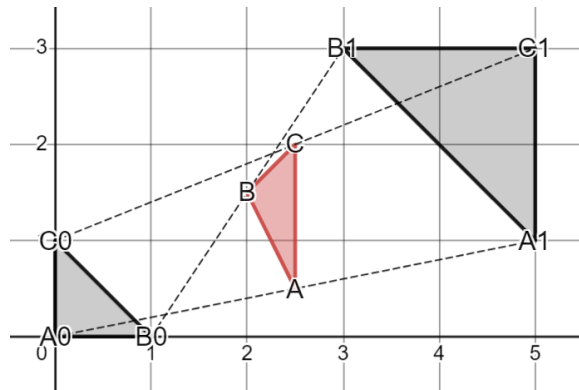
(a) Is the red triangle ever simply a line? And if so, at what time?

(b) Is the red triangle ever isosceles? The answer is yes—at $T = 0$ and at $T = 1$. But are there other times?

Triangle 2
https://www.desmos.com/calculator/fhm4x0lctw

This is a similar problem to Triangle 1, but technically more complex. So a student who has mastered Triangle 1 will still need a bit of creativity to handle Triangle 2.

Students who have been led through the construction of Triangle 1 will be happy to discover how easy it is to modify that to produce Triangle 2.

We ask the same two questions:
(a) Is the red triangle ever simply a line? And if so, at what time?

(b) Is the red triangle ever isosceles? The answer is yes—at $T = 0$ and at $T = 1$. But are there other times?
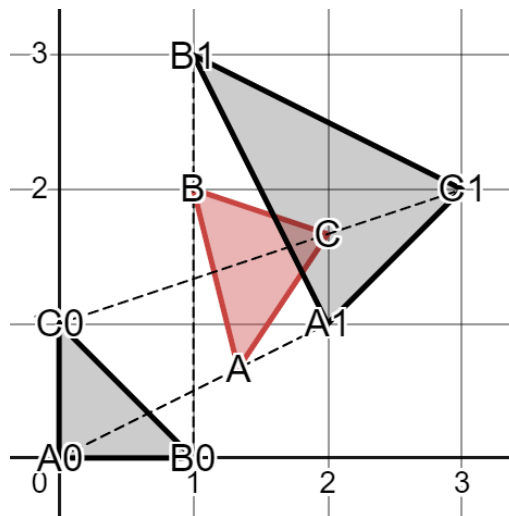
**Parametric Triangles Teacher Manual**

Quin Bouncing

Quin travels up and down the y-axis, bouncing between the line $y = 5$ and the parabola $y = x^2$. At what point does it hit the parabola?

See if you can find two different approaches to this problem. One will keep track of the number of intersections between the parabola and circle as the circle moves up and down. The other will use the fact that the slope of the parabola $y = x^2$ at any x is $2x$.

Note that you can simply show the class the Desmos animation, or you can ask them to build it themselves. Allow them to be creative in their design of a new Quin. But remember, all the embellishments have to move up and down in concert with the centre of the circle.

Quin Bouncing Teacher Manual
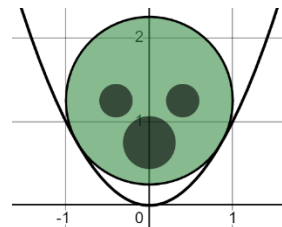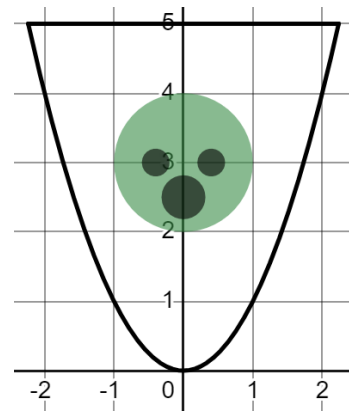
Rotating Stick

In both the grade 7 and 8 curriculum, transformations like reflections, dilations and rotations are talked about with lots of things to "learn" but not so many ideas for the students to "do." They all have calculators with sin and cos buttons so it's easy enough to simply show them how these can be used to rotate points. And once they can do that, they can rotate curves and shapes as well.

Just to emphasize the point: our objective is not to have them "understand" sin and cos, but simple to "use" sine and cos. [With the principle that understanding emerges from doing.]

First we learn how to rotate about the origin.
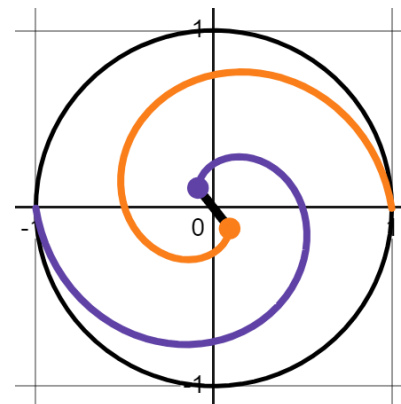
Then we discover how to simply build translation in by rotating about the origin and at the same time, moving in a continuous flow.

In one animation, we throw a rotating stick through the air so that it follows a parabolic path. At the same time, we show them how to "track" the trajectory of the endpoints of the stick.

Rotating Stick Teacher Manual

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Two Trees

A   B
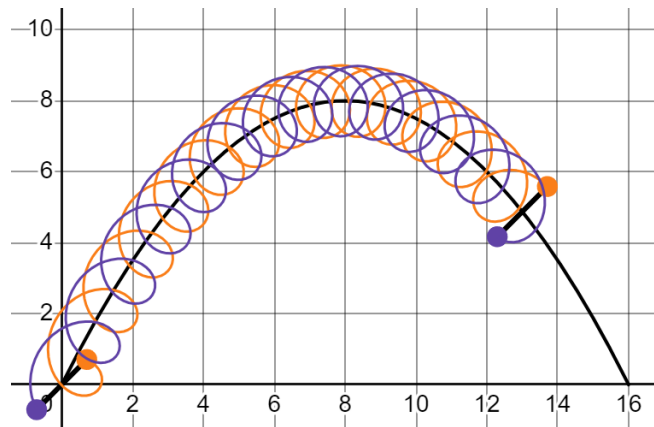
I am wandering on the grasslands and I notice two tall trees in the distance that appear to me to be exactly the same height. Now I know those trees. I call them A and B, and it is known that B is exactly twice as tall as A. To amuse myself I decide to try to walk towards the trees in such a way that they always appear to me to be the same height. What can you say about the shape of the path I will have to follow?

Two Trees Teacher Manual

Parabola Tangent

https://www.desmos.com/calculator/mtbq4rkk2f

I'm going to start with the parabola $y = x^2$ and then move it so that its vertex is at $(1, 1)$.

Then I'm going to draw a line through the origin that is tangent to the new parabola.

The question is, what is the slope of the line and what is the point of intersection? Provide your answers in the simplest possible form. You might also create some diagrams in Desmos.
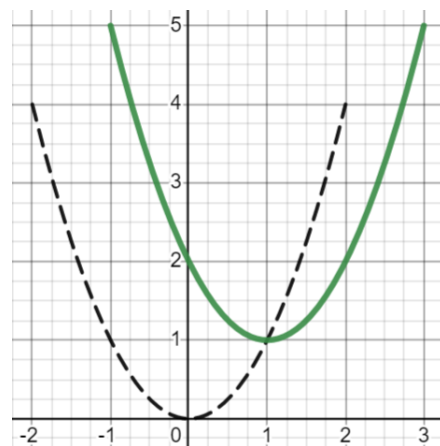
The idea is to consider the family of all lines through the origin. Some intersect the parabola twice and some don't intersect it at all. We are looking for a line that intersects it exactly once.

Parabola Tangent Teacher Manual
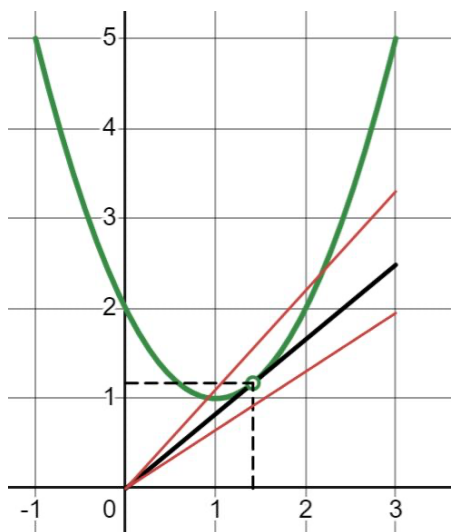
Guinea Egg

To cook a guinea egg you need to raise its temperature to $T = 90°$, but guinea eggs are delicate so this must be done slowly. There are two established methods.

The first is to put it in boiling water at a constant temperature of 100°. The second is to put it in a microwave oven at low power.

The surprising thing is that if we start with the egg at room temperature ($T = 20°$) the two methods have exactly the same cooking time–– 12 minutes.
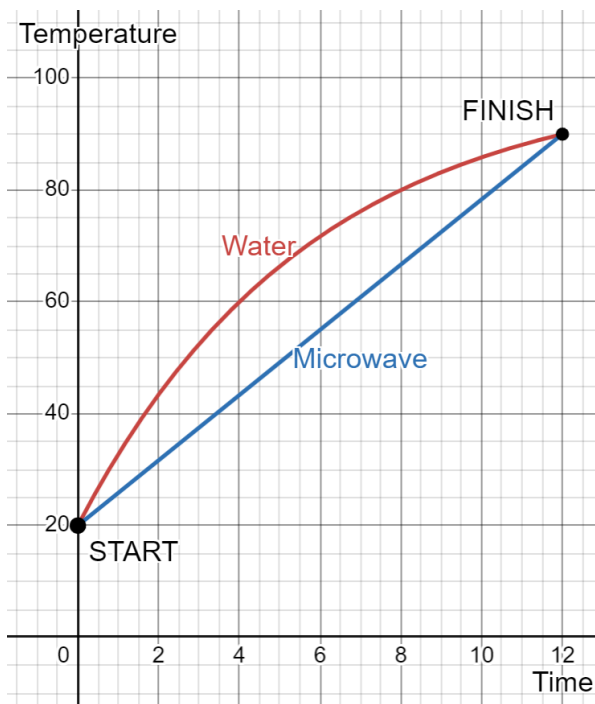
Nevertheless, the two methods employ different forms of heat transfer and this affects the shape of the temperature-time graphs, as illustrated at the right. What's going on here? This is an excellent question for the class to discuss.

How can you shorten the cooking time by moving the egg from one option to the other? Illustrate your scheme that minimizes cooking time on the graph.

The water graph follows Newton's law of temperature change and that means that it's an exponential curve. There is a property of the curve that shouts that out. Can you see it?
Hint: The temperature of the water is 100°. Compare the curve with the line $T = 100$.

Guinea Egg Teacher Manual

Missteaks

(a) Observe that
$$4\frac{4}{3} = 4 \cdot \frac{4}{3}$$

If the notation confuses you, I would say it as "4 *and* $\frac{4}{3}$ equals 4 *times* $\frac{4}{3}$." Are there other examples like this (that use integers)? Can you find them all?

(b) Here's another...
$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$$

(c) ...and another:
$$\frac{64}{16} = \frac{4}{1}$$
(cancel the 6's). Find all examples using 2-digit numbers. (We believe that there are 4.) Use teams of 4 and see which team finds them all first. Some organization pays off.

(d) This one is uber interesting:
$$2^4 = 4^2$$
$$\left(\frac{9}{4}\right)^{\frac{27}{8}} = \left(\frac{27}{8}\right)^{\frac{9}{4}}$$

Are these the start of a progression?

[**Missteaks Teacher Manual**](#)

This unit has a bit of fun and seems to work well in a Grade 9 class. It gives the students practice in working with quotient expressions, solving for one variable in terms of the others, and making simple arguments.

The Mysterious 8-cycle

Take any number n (a positive integer), square its digits, and add them up. What you get is a new number which we will call $T(n)$. T is a transformation that changes one number into another. Our objective here is to understand what happens when we iterate T.

For example, suppose we start with 2024. Then applying T again and again:

$$2024 \rightarrow 24 \rightarrow 20 \rightarrow 4 \rightarrow 16 \rightarrow 37 \rightarrow 58 \rightarrow \cdots$$

and we keep right on going. What eventually happens? Well what happens is that we have entered an interesting 8-cycle.

What happens for other starting numbers?

This is a fascinating problem. Right away the kids can start generating data and come up with a simple hypothesis. And remarkably enough, they can also prove that their conjecture is correct.
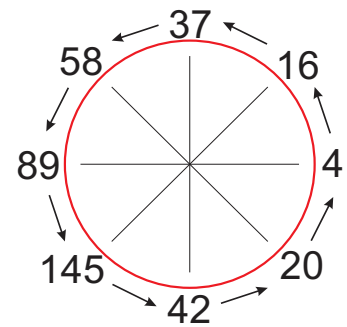
[8-Cycle Teacher Manual](#)

Red ball, Blue ball

At time $t = 0$ a red ball is projected from the origin and travels along the line $y = mx$ at constant speed 2. At the same moment a blue ball is dropped from the point $(4, 4)$ and travels down the line $x = 4$ at constant speed 1.

(a) The problem is to find the value of m (the slope of the red line) that will produce a direct collision of the two balls. We want you to do this by using the Pythagorean Theorem to find a quadratic equation that m must satisfy. Then you can use the quadratic formula to solve for m, or simply plot it on Desmos and get a good approximation of its roots.

(b) Use Desmos to actually construct the animation and check your answer in (a). Design an interesting display to fire when the collision occurs (or is very close to occurring).

This is a great problem and perfect for grade 9. We have certainly used it in a grade 8 class who have the Pythagorean Theorem.

The (b) part, of programming the action on Desmos, turns out to be more challenging than we expected. The students have no problem moving the ball down the blue line at speed 1. But moving it up the line of slope m at speed 2, is more difficult.
https://www.desmos.com/calculator/kibcchclxo

**Red ball Blue ball Teacher Manual**
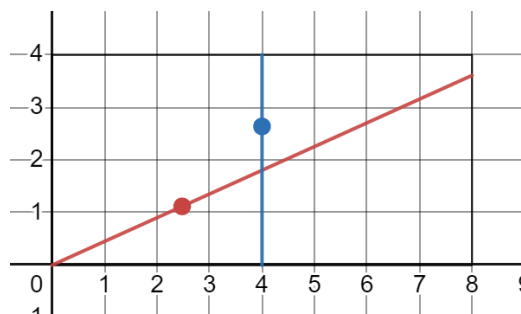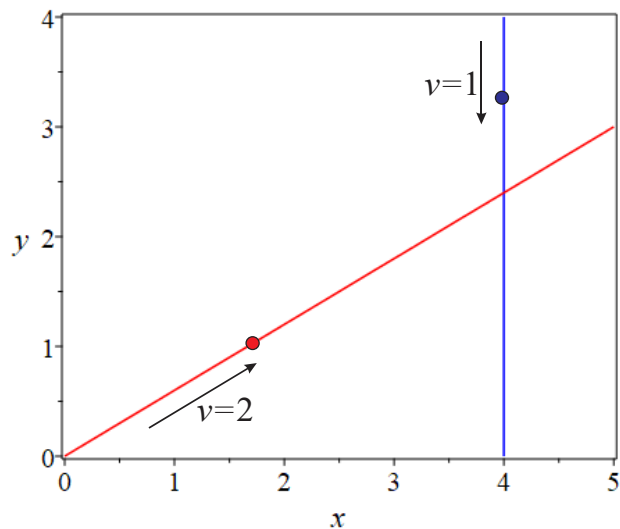
Mirrors

You are in the middle of two mirrors set at an angle of 50° and your eyes are 1 meter from the vertex. You can see 6 images of yourself. Here's the problem:

(a) Plot, for each image, the path of the light ray as it travels from your nose back to your eye.
(b) Calculate, for each image, its apparent distance from you.

In our view, this problem is a "must" for grade 10. It's challenging—there's no doubt about that––but it is experiential with a capital E. This is a beautiful intersection of triangle trig with the law of reflection––too good to miss.

And at the end, as you will see, there's an extraordinarily simple formula that derives from an elegant argument.

Suppose you look at your nose. Then that light-ray must start at your nose and end at your eyes. Strangely enough, that's a hard idea for them to get hold of. Others draw paths that go outside the triangle of the mirrors, but that certainly cannot happen. Then someone discovers that some images are left-right reversed and others aren't. *What's with that*?! The students go back to the mirrors to check.
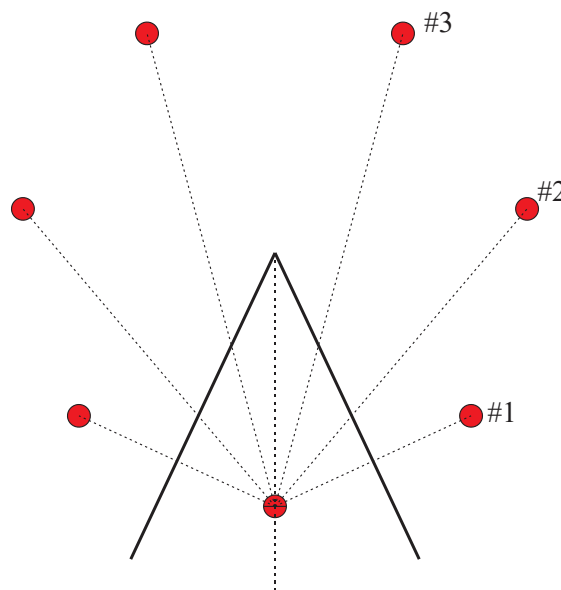
Mike, looking into the mirror at image #3. (Photo credit Siobhain)

We bought these mirrors in JYSK for $12 each, used duct tape to put them together, and set them up in the classroom. It was worth it—the kids kept going back to look and think.

Mirrors Teacher Manual
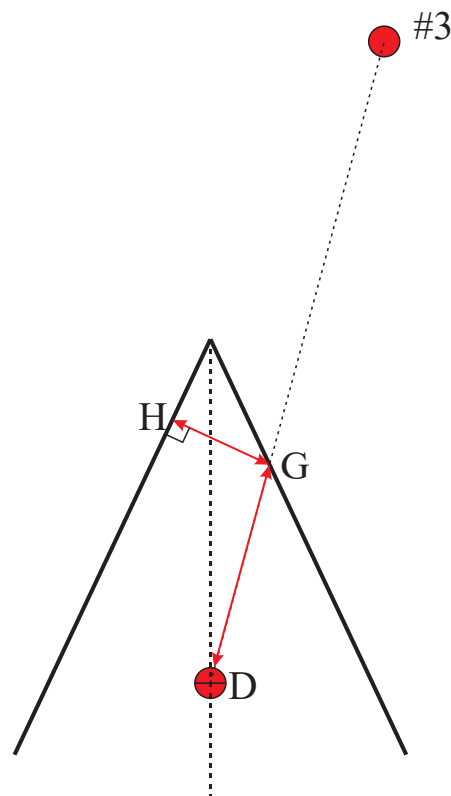
Trains

Trains are constructed by putting together two kinds of cars, those of length 1 and those of length 2. The problem is to find out how many different trains there are of total length 12 units

For example, here is one possibility, using 6 cars of length 1 and 3 cars of length 2.

1	1	1	2	2	1	2	1	1

Other possibilities are obtained by rearranging the cars, or using different numbers of short and long cars.

It's hard to know how to start thinking about this. We could try to list all the possibilities in some systematic order, but there are a large number of trains of length 12, and we hope there might be a better way.

One idea is to look at trains that are much shorter, find out how many there are by direct counting, and see if we can find a pattern. For example the table at the right shows that there are 8 different trains of length 5.

To have some notation, let t_n denote the number of trains of length n. Then the table tells us that

$$t_5 = 8.$$

When work out the answer for the other small numbers we get a striking pattern: the Fibonacci numbers. If we assume that this pattern continues to hold, we get our answer:

$$t_{12} = 233.$$

But does it continue to hold? The best way to tackle that question is to assume that it *does* hold and ask why. Why should the number of trains follow the Fibonacci law:

$$t_{n+1} = t_n + t_{n-1}?$$

That's the first question we need to answer. But if we can do that, we will have in our hands a powerful new tool for: a "trains method" for establishing some of the many remarkable numerical properties of the Fibonacci numbers.

For example: $5^2 + 8^2 = 89$. Find others like this. Why does it hold? There's a remarkable "trains" argument for this.

[Trains Teacher Manual](#)

Here are the building blocks: cars of length 1 and cars of length 2.
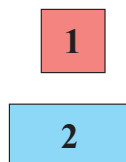
For example, here are the 8 trains of length 5
1-1-1-1-1
1-1-1-2
1-1-2-1
1-2-1-1
2-1-1-1
1-2-2
2-1-2
2-2-1

n	t_n
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233

Hitting 10

You toss a coin repeatedly. If heads is worth 1 and tails is worth 2, and you add up the score as you go along, what is the probability that you'll hit 10? That is, if you announce your new total after every toss, what is the probability you will get to announce 10?

This is a good problem to follow *Trains.* Students can start by collecting data, either with real coins or dice or with technology.

Then we can do the same thing we did with *Trains.* Calculate the probabilities for small numbers (like hitting 5) by finding all the cases, being careful with the probabilities. For example 1121 has a different probability than 122.

Now look for a recursive relationship. If you know the probabilities for $N = 8$ and $N = 9$. Can you work out from those the probability for $N = 10$?

In fact they should be able to argue that

$$p_{10} = \frac{1}{2}p_9 + \frac{1}{2}p_8.$$

and that provides for them a guaranteed calculation of p_{10}. They might even have guessed this relationship from the probabilities they already calculated, but without any certainty.

There are other interesting arguments as well. For example, if I only give you p_9, there is simple formula for p_{10} in terms of that. Can you find that?

Hitting 10 Teacher Manual

This problem can be engaging at any level between grade 8 and first-year university.

Target N	Probability p_N
1	1/2
2	3/4
3	5/8
4	11/16
5	21/32
6	
7	
8	

| Target N	Probability p_N
1	1/2
2	3/4
3	5/8
4	11/16
5	21/32
6	43/64
7	85/128
8	171/256
9	341/512
10	683/1024

Tournament

A, B and C are playing in the simple tournament diagrammed at the right. B and C play first and then the winner plays A.

Data from past matches give us the following predictions:

If A plays B: A will win 70% of the time
If A plays C: A will win 80% of the time
If B plays C: B will win 60% of the time

Based on these data, calculate the probability A will win the tournament. Use an effective diagram to illustrate your argument.
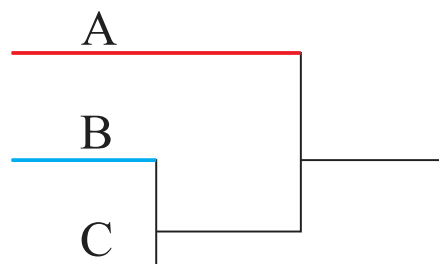
Tournament Teacher Manual

Wondercards & School Coffee

Part 1: Wondercards are either red or blue, and each card is either happy or sad.
- 40% of all cards are red.
- 60% of all cards are blue.
- 3/4 of all red cards are happy.
- 1/3 of all blue cards are happy.

I choose a random card and it turns out to be happy. What is the probability that it is red?

Part 2: Students in their spare period often visit a coffee house that is close to the school. During one period the manager kept track of coffee sales to students and reported that there were twice as many girls as boys.

But when the results of the survey were reported in the student paper, the headline read: "Girls Twice as Likely as Boys to Drink Coffee."

Are these two assertions equivalent? [Note that we assume all coffee drinkers buy their coffee at the coffee house.]

WonderCards Teacher Manual.

Greedy

The investment
To start with you choose any number *x* between 0 and 1. We'll call *x* your "investment." There is a reason your investment should be large—the higher the investment, the higher will be the payoff each time you play the game. But there is also a reason your investment should be small—the higher the investment, the more likely you are to have to leave the games after each round. Somewhere in the middle there ought to be an optimal investment.

The payoff.
Each time you play the game you'll get a payoff $y = f(x)$, whose graph appears at the right. If you invest less than 0.2 the payoff is zero. For $x > 0.2$, the payoff increases with *x* following the graph. For example, If you invest $x = 0.4$, you will get a payoff of $y = 0.5$ with each play of the game.
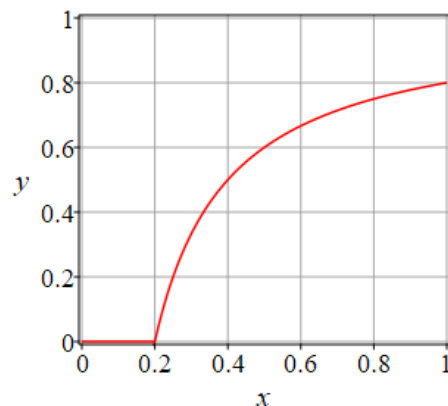
The z-effect
After each play of the game, a random number *z* is generated between 0 and 1. If *z* is greater than your investment *x*, you get to play again and get another payoff of $f(x)$. Following that, another random number *z* between 0 and 1 is generated and if it is greater than *x*, you play again, etc. Your winnings accumulate over successive plays.
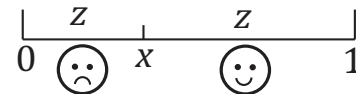
The game goes on like this until the random number *z* is less than *x*. At that point the game is over and you take your total winnings home.

The question of course is to find the value of *x* that will maximize your expected winnings. See if you can find both an algebraic and graphical approach to this problem.

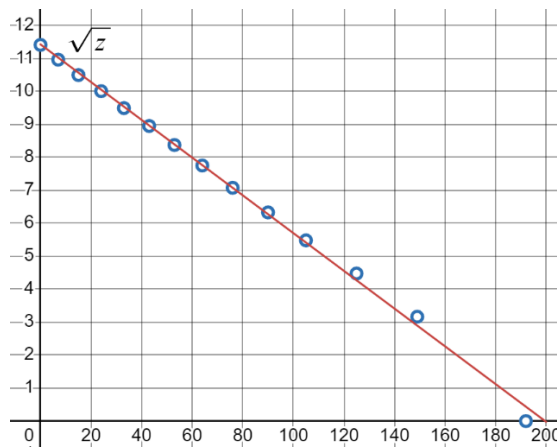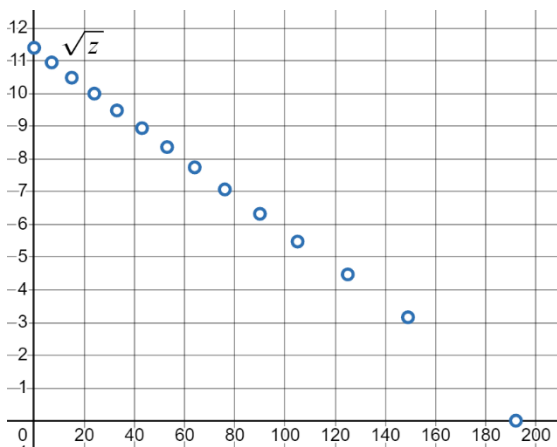**Greedy Teacher Manual**

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WaterTank

A cylindrical bottle is filled to a level of 130 mm. At time $t=0$ (minutes) a hole is punched in the bottom, and water begins to flow out. It takes exactly 192 seconds for the tank to empty. We instruct the class to record the times at which the water arrives at each of the marked levels and plot these ponts on a set of t-z axes.

https://www.desmos.com/calculator/ik6czhzwax
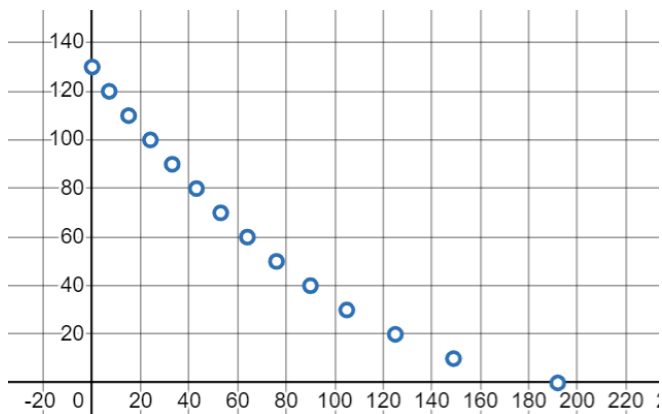
We now use these data to embark upon a fascinating journey of discovery. First of all, we note that Newton's laws predict that the points ought to lie along a parabola. Given that we find that the "best-fit" parabola does pass pretty well through the points.

As a visual check we find a way to transform the data into a set of points that ought to lie along a line—essentially we plot $\sqrt{z}$ against t. That linear set of points is quite a sight.


But when we actually add the "best-fit" line we find an interesting discrepancy towards the end of the process, and tracking *that one* down teaches us something about viscosity.

WaterTank Teacher Manual

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Two Towns

This is a remarkable investigation. You could spend a week on this, and the students would learn so much. It's an interesting and sophisticated and often surprising "maker space" particularly with the Desmos interface we have provided. It fits perfectly with our ambitions for a curriculum rich with design and construction.

What the students are asked to do is move back and forth between the standard *x-y* coordinate system and a second system which in a sense is a distorted mirror of the first.
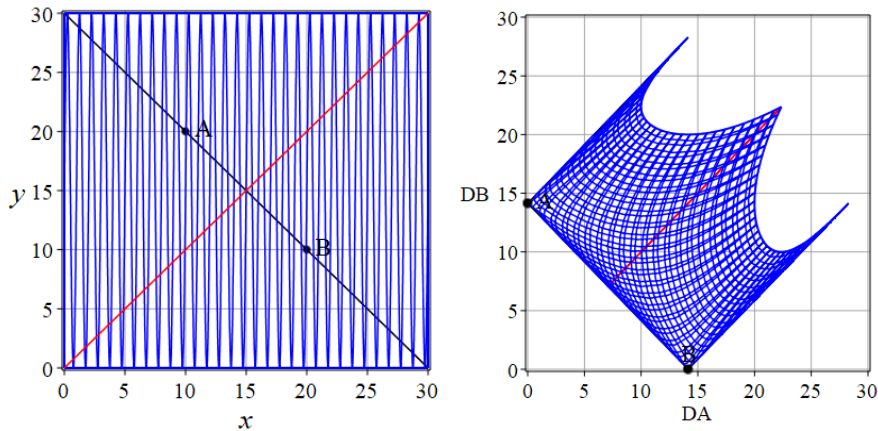
We've called it "sophisticated" but that doesn't make it technically difficult. However, it does require careful, precise thinking.

As an example, study the two graphs below. On the left is a landscape graph embroidered with two lines and three circles, and on the right is its corresponding distance graph, its distorted mirror.
https://www.desmos.com/calculator/3kg7l1xade
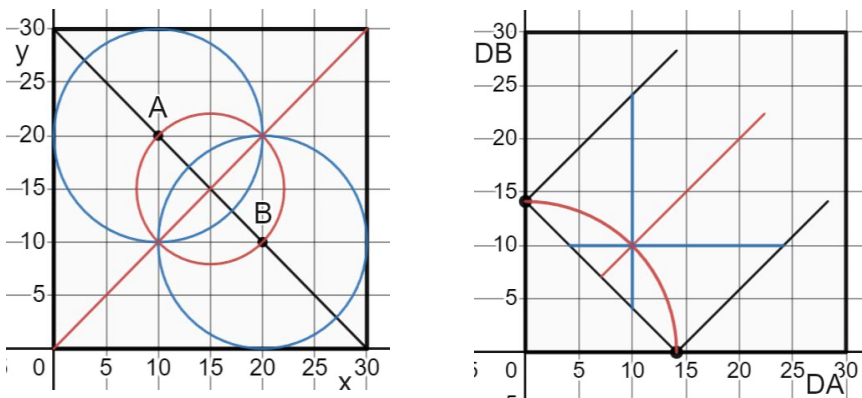
This is what we give to the students. And then they play the following Desmos game. They take any line or curve they want in the landscape graph (think of it as a road along which you decide to travel) and construct its corresponding locus in the distance graph, a curve that keeps track of its distance from each of the towns.

Two Towns Teacher Manual
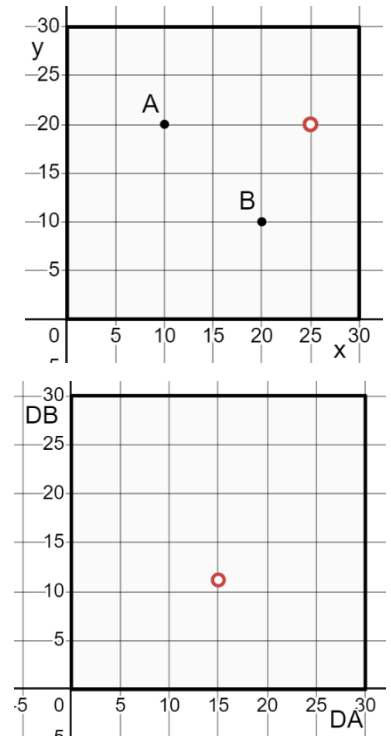
The first graph above is the landscape graph and the second is the distance graph. The landscape has two towns, at points A and B. A red point has been plotted on this graph. The coordinates of its "image" on the distance graph are its distance DA from A and its distance DB from B.

Study the image above and be sure you understand the point in the DA-DB graph.

Anyway, that's the game we will be playing here. Every line of curve has a corresponding line or curve on the distance graph and the "map" is illustrated by the two small red circles above.